

# FAKE LIFTINGS OF GALOIS COVERS BETWEEN SMOOTH CURVES

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**ABSTRACT.** In this paper we investigate the problem of lifting of Galois covers between algebraic curves from characteristic  $p > 0$  to characteristic 0. We prove a refined version of the main result of Garuti concerning this problem in [Ga]. We formulate a refined version of the Oort conjecture on liftings of cyclic Galois covers between curves. We introduce the notion of fake liftings of cyclic Galois covers between curves, their existence would contradict the Oort conjecture, and we study the geometry of their semi-stable models. Finally, we introduce and investigate on some examples the smoothening process, which ultimately aims to show that fake liftings do not exist. This in turn would imply the Oort conjecture.

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**§0. Introduction.** In this paper we investigate the following problem, known as the problem of lifting of Galois covers between algebraic curves from characteristic  $p > 0$  to characteristic 0.

In what follows  $R$  will denote a complete discrete valuation ring of unequal characteristics,  $K \stackrel{\text{def}}{=} \text{Fr}(R)$  the quotient field of  $R$ ,  $\text{char}(K) = 0$ , and  $k$  the residue field of  $R$ , which we assume to be algebraically closed of characteristic  $p > 0$ .

**Problem I.** Let

$$f_k : Y_k \rightarrow X_k$$

be a finite Galois cover between smooth  $k$ -curves, with Galois group  $G$ . Is it possible to lift the Galois cover  $f_k$  to a Galois cover

$$f : Y' \rightarrow X'$$

between smooth  $R'$ -curves, where  $R'/R$  is a finite extension?

In the original version of this problem, one doesn't fix  $R$ , but fixes  $k$ ,  $f_k : Y_k \rightarrow X_k$ , and asks for the existence of a local domain  $R$  dominating the ring of Witt vectors  $W(k)$  over which a lifting of  $f_k$  exists as part of the problem (cf. [Oo]).

One can formulate Problem I in terms of lifting of curves and their automorphism groups from positive to zero characteristics (cf. [Oo], [Oo1]). One can also formulate the following variant of the above problem, where one fixes a lifting of the curve  $X_k$ .

**Problem II.** Let  $X$  be a proper, smooth, geometrically connected  $R$ -curve, and

$$f_k : Y_k \rightarrow X_k \stackrel{\text{def}}{=} X \times_R k$$

a finite Galois cover between smooth  $k$ -curves, with Galois group  $G$ . Is it possible to lift the Galois cover  $f_k$  to a Galois cover

$$f : Y \rightarrow X' \stackrel{\text{def}}{=} X \times_R R'$$

where  $R'/R$  is a finite extension, and  $Y$  is a smooth  $R'$ -curve?

We shall refer to a lifting  $f$  as above, if it exists, as a smooth lifting of the Galois cover  $f_k$ .

The above two problems are in fact equivalent (cf. discussion in 3.1), and one may well consider Problem II, instead.

This problem has been considered successfully by Grothendieck in the case where  $f_k$  is a tamely ramified cover. In this case a smooth lifting  $f$  as above exists over  $R$  (cf. [Gr]).

The answer to this problem is however No in general. Indeed, in the case where  $G$  is the full automorphism group of  $Y_k$ , there are examples where the size of  $G$  exceeds the Hurwitz bound for the size of automorphism groups of curves in characteristic zero (cf. [Ro]), and the cover  $f_k$  can not be lifted in this case. Also it is in general necessary to perform a finite extension of  $R$  in order to solve this problem (cf. [Oo], 1).

In the case where  $f_k$  is wildly ramified there are non liftable examples with Galois groups as simple as  $G \xrightarrow{\sim} \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$  (cf. [Gr-Ma], 5). See also [Oo], 1, for an example of a genus 2 curve in characteristic 5, and an automorphism group of cardinality 20, which cannot lift to characteristic 0.

The most general result one can hope for, in the case where  $f_k$  is wildly ramified, is the following which was conjectured by F. Oort.

**Oort conjecture [Conj-O].** Problem I, or equivalently Problem II, has a positive answer if  $G \xrightarrow{\sim} \mathbb{Z}/m\mathbb{Z}$  is a cyclic group. Moreover, in this case one can choose  $R'$  in Problem I, and Problem II, to be the minimal extension of  $R$  which contains the  $m$ -th roots of 1.

In order to solve this conjecture one may reduce to the case where  $G \xrightarrow{\sim} \mathbb{Z}/p^n\mathbb{Z}$  is a cyclic  $p$ -group (cf. Lemma 3.1.1). In this case Oort conjecture has been verified when  $n \leq 2$  (cf. [Se-Oo-Su] for the case  $n = 1$ , and [Gr-Ma] for the case  $n = 2$ ). In the approach of Oort, Sekiguchi, Suwa, Green, and Matignon one uses the Oort-Sekiguchi-Suwa theory, which provides [explicit] equations describing the degeneration of the Kummer equations in characteristic 0 to the Artin-Schreier-Witt equations in characteristic  $p > 0$ . In general, this conjecture is still widely open. To the best knowledge of the author, no concrete liftable examples are known in the case where  $n \geq 3$ .

However, if in Problem II one relaxes the requirement that  $Y$  in the lifting  $f : Y \rightarrow X' \stackrel{\text{def}}{=} X \times_R R'$  is smooth over  $R'$ , one has the following rather general result where one allows introducing singularities in  $Y_k$ , and which is due to Garuti (cf. [Ga], and Theorem 2.5.1).

**Theorem A (Garuti).** *There exists a finite extension  $R'/R$ , and a finite Galois cover  $f' : Y' \rightarrow X' \stackrel{\text{def}}{=} X \times_R R'$ , with Galois group  $G$ , where  $Y'$  is a normal  $R'$ -curve (which in general need not be smooth over  $R'$ ), and the natural morphism  $f'_k : Y'_k \stackrel{\text{def}}{=} Y' \times_{R'} k \rightarrow X_k$  between special fibres is generically Galois with Galois group  $G$ . Moreover, there exists a factorisation  $f_k : Y_k \xrightarrow{\nu} Y'_k \xrightarrow{f'_k} X_k$ , where the morphism  $\nu : Y_k \rightarrow Y'_k$  is a morphism of normalisation, which is an isomorphism outside the ramified points, and  $Y'_k$  is unibranch.*

We call  $f'$  as in Theorem A a Garuti lifting of the Galois cover  $f_k$ .

In the first part of this paper, in §2, we revisit Garuti's theory. We prove the following refined version of Theorem A (cf. Theorem 2.5.3).

**Theorem B.** *We use the same notations as in Theorem A. Let  $H$  be a quotient of  $G$ , and  $g_k : Z_k \rightarrow X_k$  the corresponding Galois sub-cover of  $f_k$  with Galois group  $H$ . Let  $h' : Z' \rightarrow X' \stackrel{\text{def}}{=} X \times_R R'$  be a Garuti lifting of the Galois cover  $h_k$ , defined over the finite extension  $R'/R$ . Then there exists a finite extension  $R''/R'$ , and a Garuti lifting  $f'' : Y'' \rightarrow X'' \stackrel{\text{def}}{=} X \times_R R''$  of the Galois cover  $f_k$  over  $R''$ , which dominates  $h'$ , i.e. we have a factorisation  $f'' : Y'' \xrightarrow{g''} Z'' \stackrel{\text{def}}{=} Z' \times_{R'} R'' \xrightarrow{h'' \stackrel{\text{def}}{=} h' \times_{R'} R''} X''$ , where  $g'' : Y'' \rightarrow Z''$  is a finite morphism between normal  $R''$ -curves.*

In the course of proving this result we prove a structure theorem concerning a certain quotient of the "geometric Galois group" of a  $p$ -adic open disc, which is the most relevant to the lifting problem. This result might be of interest independently from the lifting problem.

Let  $\tilde{X} \stackrel{\text{def}}{=} \text{Spf } R[[T]]$ ,  $\tilde{X}_K \stackrel{\text{def}}{=} \text{Spec}(R[[T]] \times_R K)$  [a  $p$ -adic open disc over  $K$ ], and  $\mathcal{X} \stackrel{\text{def}}{=} \text{Spf } R[[T]]\{T^{-1}\}$  the formal boundary of  $\tilde{X}$  (cf. §2). Let  $\Delta$  (resp.  $\Delta'$ ) be the maximal pro- $p$  group which classifies geometric Galois covers of  $\tilde{X}$  (resp. of  $\mathcal{X}$ ) which are pro- $p$ , and which are generically étale at the level of special fibres (cf. 2.3, and 2.4, for more precise definitions).

**Theorem C.** (cf. Theorem 2.3.1, and Theorem 2.4.1) *The profinite group  $\Delta$  is a free pro- $p$  group. Moreover, there exists a natural morphism  $\Delta' \rightarrow \Delta$  which makes  $\Delta'$  into a direct factor  $\Delta$  (cf. 1.1, for the definition, and characterisation, of a direct factor of a free pro- $p$  group).*

In light of Theorem B, we revisit in §3, 3.1, the Oort conjecture. We formulate the following refined version of this conjecture (cf. 3.1, for more details).

**Oort Conjecture Revisited [Conj-O-Rev].** We use the same notations as in Problem II. Assume that  $G \xrightarrow{\sim} \mathbb{Z}/m\mathbb{Z}$  is a cyclic group. Let  $H$  be a quotient of  $G$ , and  $g_k : Z_k \rightarrow X_k$  the Galois sub-cover of  $f_k$  with Galois group  $H$ . Then there exists a smooth Galois lifting  $g : Z' \rightarrow X' \stackrel{\text{def}}{=} X \times_R R'$  of  $g_k$  over some finite extension  $R'/R$ .

Furthermore, for every smooth lifting  $g$  of the Galois sub-cover  $g_k$  of  $f_k$  as above, there exists a smooth lifting  $f : Y'' \rightarrow X'' \stackrel{\text{def}}{=} X \times_R R''$  of  $f_k$ , over some finite extension  $R''/R'$ , such that  $f$  dominates  $g$ , i.e. we have a factorisation  $f : Y'' \rightarrow Z'' \stackrel{\text{def}}{=} Z' \times_{R'} R'' \xrightarrow{g \times_{R'} R''} X''$ . Moreover,  $R''$  can be chosen to be the minimal extension of  $R'$  which contains a primitive  $m$ -th root of 1.

As for the original Oort conjecture, to prove this revisited version one may reduce to the case where  $G \xrightarrow{\sim} \mathbb{Z}/p^n\mathbb{Z}$ . In the case  $n = 1$ , both [Conj-O] and [Conj-O-Rev] are clearly equivalent. In 3.2, and in the case where  $n = 2$ , we verify [Conj-O-Rev] in some cases (cf. Lemma 3.2.1, and Lemma 3.2.2).

The second main part of this paper is motivated by the idea of the search for a path, or a bridge, between Garuti's theory and the (revisited) Oort conjecture, which may lead to the solution of this conjecture. We introduce in §3 the notion of fake liftings of cyclic Galois covers between curves, with the purpose of establishing such a bridge.

Next, we explain the definition of fake liftings, and the simple idea which leads to their existence.

Assume that  $G \xrightarrow{\sim} \mathbb{Z}/p^n\mathbb{Z}$ ,  $n \geq 1$ . Let  $H$  be the unique quotient of  $G$  with cardinality  $p^{n-1}$ . We use the notations in Problem II, and assume that  $X = \mathbb{P}_R^1$ . In fact one can reduce the solution of Problem II to this case, (cf. 3.1, and Lemma 3.1.1)

Let  $f_k : Y_k \rightarrow \mathbb{P}_k^1$  be a finite ramified Galois cover with Galois group  $G$ , and  $g_k : X_k \rightarrow \mathbb{P}_K^1$  the Galois sub-cover of  $f_k$  with Galois group  $H$ . In order to solve [Conj-O-Rev] for the Galois cover  $f_k$ , and the sub-cover  $g_k$ , one may proceed by induction on the cardinality of the group  $G$ . The case where  $G$  has cardinality  $p$  is solved in [Se-Oo-Su].

So we may assume, by an induction hypothesis, that  $g_k$  admits a smooth lifting  $g : \mathcal{X} \rightarrow \mathbb{P}_R^1$  defined over  $R$ , where  $\mathcal{X}$  is a smooth  $R$ -curve, i.e. we assume that [Conj-O] holds for the Galois sub-cover  $g_k$  of  $f_k$ . We would like to show that [Conj-O-Rev] is true for  $f_k$ , and the smooth lifting  $g$  of the sub-cover  $g_k$ , i.e. show that  $g$  can be dominated by a smooth lifting of  $f_k$ , after eventually a finite extension of  $R$ .

Consider all possible Garuti liftings  $f : \mathcal{Y} \rightarrow \mathcal{X} \xrightarrow{g} \mathbb{P}_R^1$  of  $f_k$ , which dominate the smooth lifting  $g$  of  $g_k$ . These Garuti liftings exist by the above refined version of Garuti's theory, in Theorem B, and are a priori defined over a finite extension of  $R$ . For a Garuti lifting  $f$  as above, which we can suppose defined over  $R$  without loss of generality, the degree of the different in the morphism  $f_K : \mathcal{Y}_K \stackrel{\text{def}}{=} \mathcal{Y} \times_R K \rightarrow \mathbb{P}_K^1$  between generic fibres is greater than the degree of the different in the morphism  $f_k : Y_k \rightarrow \mathbb{P}_k^1$ . Moreover,  $\mathcal{Y}$  is smooth over  $R$ , which implies that [Conj-O-Rev] holds in this case, if and only if these degrees of different are equal.

Next, we argue by contradiction. Assume that [Conj-O-Rev] doesn't hold for the Galois cover  $f_k$ , and the smooth lifting  $g$  of the sub-cover  $g_k$ . In particular, for all possible Garuti liftings  $f$  as above,  $\mathcal{Y}$  is not smooth over  $R$ . A Garuti lifting  $f : \mathcal{Y} \rightarrow \mathbb{P}_R^1$  as above such that the degree of the different in the morphism  $f_K : \mathcal{Y}_K \stackrel{\text{def}}{=} \mathcal{Y} \times_R K \rightarrow \mathbb{P}_K^1$  between generic fibres is minimal among all possible  $f$ 's, is called a fake lifting of the Galois cover  $f_k$ , relative to the smooth lifting  $g$  of  $g_k$  (cf. Definition 3.3.2).

Fake liftings won't exist if [Conj-O-Rev] is true. In fact in order to prove [Conj-O-Rev] for the Galois cover  $f_k$ , relative to the smooth lifting  $g$  of  $g_k$ , it suffices to show that fake liftings  $f$  as above do not exist (cf. Remark 3.3.3).

One expects fake liftings to have very special properties, which eventually may lead to their non existence. Special properties of fake liftings should be encoded in their semi-stable models.

Let  $f : \mathcal{Y} \rightarrow \mathcal{X} \xrightarrow{g} \mathbb{P}_R^1$  be a fake lifting as above, assuming it exists. In §3 we

study the geometry of a minimal semi-stable model  $\mathcal{Y}' \rightarrow \mathcal{Y}$  of  $\mathcal{Y}$ , which we suppose defined over  $R$ , and in which the ramified points in the morphism  $f_K : \mathcal{Y}_K \rightarrow \mathbb{P}_K^1$  specialise in smooth distinct points of  $\mathcal{Y}'_k \stackrel{\text{def}}{=} \mathcal{Y}' \times_R k$ . It turns out that these semi-stable models have indeed very specific properties, which are in some sense reminiscent to the properties of the minimal semi-stable models of smooth liftings of cyclic Galois covers between curves.

We prove, among other facts, that the configuration of the special fibre  $\mathcal{Y}'_k \stackrel{\text{def}}{=} \mathcal{Y}' \times_R k$  of the semi-stable model  $\mathcal{Y}'$ , of the fake lifting  $f$ , is a tree-like (cf. Theorem 3.5.4). Moreover, all the irreducible components of positive genus in  $\mathcal{Y}'_k$ , and which contribute to the difference between the generic and special different in the morphism  $f : \mathcal{Y} \rightarrow \mathbb{P}_R^1$ , are end vertices of the tree associated to  $\mathcal{Y}'_k$  with special properties (cf. loc. cit). In the course of proving this result we establish some of the properties of the minimal semi-stable model of an order  $p^n$  automorphism of a  $p$ -adic open disc, with no inertia at the level of special fibres, that were established in the case  $n = 1$  in [Gr-Ma1] (cf. Proposition 3.5.3).

Finally, in §4, we introduce the smoothening process for a fake lifting  $f : \mathcal{Y} \rightarrow \mathcal{X} \xrightarrow{g} \mathbb{P}_R^1$  as above. The ultimate aim of this process is to show that fake liftings do not exist. This in turn would prove [Conj-O-Rev].

The basic idea of smoothening of the fake lifting  $f$  is to construct, starting from  $f$ , a new Garuti lifting  $f_1 : \mathcal{Y}_1 \rightarrow \mathcal{X} \xrightarrow{g} \mathbb{P}_R^1$ , which dominates the smooth lifting  $g$  of  $g_k$ , and such that the degree of the different in the morphism  $f_{1,K} : \mathcal{Y}_{1,K} \stackrel{\text{def}}{=} \mathcal{Y}_1 \times_R K \rightarrow \mathbb{P}_K^1$  between generic fibres is smaller than the degree of the different in the morphism  $f_K : \mathcal{Y}_K \stackrel{\text{def}}{=} \mathcal{Y}' \times_R K \rightarrow \mathbb{P}_K^1$ . We call such  $f_1$  a smoothening of  $f$ . If this construction is possible, it would imply that the fake lifting  $f$  doesn't exist. Indeed, this would contradict the minimality of the generic different in  $f$ . Hence this will prove [Conj-O-Rev] for the Galois cover  $f_k$ , and the smooth lifting  $g$  of the sub-cover  $g_k$ .

We describe a formal way, using formal patching techniques, to construct a smoothening  $f_1$  of the fake lifting  $f : \mathcal{Y} \rightarrow \mathbb{P}_R^1$ , starting from the minimal semi-stable model  $\mathcal{Y}' \rightarrow \mathcal{Y}$  of  $\mathcal{Y}$  (cf. 4.1). This construction is related to the existence of (internal) irreducible components in the special fibre  $\mathcal{P}_k$  of the quotient  $\mathcal{P} \stackrel{\text{def}}{=} \mathcal{Y}'/G$  of the semi-stable model  $\mathcal{Y}'$  by  $G$ , which satisfy certain technical conditions arising from the geometry of the semi-stable model  $\mathcal{Y}'$ , and the Galois cover  $\mathcal{Y}' \rightarrow \mathcal{P}$ . We call such a component a removable vertex of the tree associated to  $\mathcal{P}_k \stackrel{\text{def}}{=} \mathcal{P} \times_R k$  (cf. Definition 4.1.2). The existence of a removable vertex in  $\mathcal{P}_k$  leads immediately to the existence of a smoothening  $f_1$  of the fake lifting  $f$  as above (cf. Definition 4.1.3).

We show that the smoothening process is possible in the case where  $G \xrightarrow{\sim} \mathbb{Z}/p\mathbb{Z}$ , i.e.  $n = 1$  (cf Proposition 4.2.2). This gives an alternative proof of the Oort conjecture in this case. This proof, though simple, is striking in the view of the author in many respects.

First, this proof is not explicit, in the sense that it doesn't produce an explicit lifting of the Galois cover  $f_k$ . It doesn't even produce any automorphism of order  $p$  of a  $p$ -adic open disc. Second, the proof doesn't rely (in any form) on the degeneration of the Kummer equation to the Artin-Schreier equation as in [Se-Oo-Su] (cf. also [Gr-Ma]), but rather on the degeneration of the Kummer equation to a radical equation (cf. proof of Proposition 4.2.2). This suggests the possibility of proving

[Conj-O] without using the Oort-Sekiguchi-Suwa theory.

In the case where  $n = 2$ , i.e.  $G \xrightarrow{\sim} \mathbb{Z}/p^2\mathbb{Z}$ , we give, in 4.3, some sufficient conditions for the existence of removable vertices which lead to the execution of the smoothening process (cf. Proposition 4.3.1).

Next, we briefly review the content of each section of this paper. In §1 we collect some background material which is used in this paper. This includes background material on pro- $p$  groups, and formal patching techniques. In §2 we revisit the theory of Garuti. We prove Theorem B (Theorem 2.5.3), and our main result Theorem C concerning the structure of a certain quotient of the geometric Galois group of a  $p$ -adic open disc (Theorem 2.3.1, and Theorem 2.4.1). Both §1 and §2 can be read independently of the rest of the paper. In §3 we revisit Oort conjecture, and introduce the notion of fake liftings of cyclic Galois covers between curves. We then establish the main properties of their minimal semi-stable models in Theorem 3.5.4. In §4 we introduce the notion of the smoothening process for fake liftings, and we investigate on some examples, in degree  $p$  and  $p^2$ , this process. This section is dependent on §3.

**§1 Background.** In this section, and for the convenience of the reader, we collect some background material which is used in this paper. We recall some well-known facts on pro- $p$  groups, that will be used in §2 (cf. 1.1). We state the main result of formal patching that we use in this paper, and which plays a crucial role in the proof of the main results in §2, §3, and §4 (cf. 1.2). Finally, we recall the degeneration of  $\mu_p$ -torsors above boundaries of formal fibres at closed points of formal curves (cf. 1.3).

**1.1 Complements on pro- $p$  Groups.** In this sub-section we fix a prime integer  $p > 1$ . We recall some well-known facts on profinite pro- $p$  groups that will be used in §2.

First, we recall the following characterisations of free pro- $p$  groups.

**Proposition 1.1.1.** *Let  $G$  be a profinite pro- $p$  group. Consider the following properties:*

- (i)  $G$  is a free pro- $p$  group.
- (ii) The  $p$ -cohomological dimension of  $G$  satisfies  $\mathrm{cd}_p(G) \leq 1$ .
- (iii) Given a surjective homomorphism  $\sigma : Q \twoheadrightarrow P$  between finite  $p$ -groups, and a continuous surjective homomorphism  $\phi : G \twoheadrightarrow P$ , there exists a continuous homomorphism  $\psi : G \rightarrow Q$  such that the following diagram is commutative:

$$\begin{array}{ccc} G & \xrightarrow{\mathrm{id}} & G \\ \psi \downarrow & & \downarrow \phi \\ Q & \xrightarrow{\sigma} & P \end{array}$$

*Then the following equivalences hold:*

$$(i) \iff (ii) \iff (iii).$$

*Proof.* Well-known (cf. [Se], and [Ri-Za], Theorem 7.7.4).  $\square$

Next, we recall the notion of a direct factor of a free pro- $p$  group (cf. [Ga], 1, the discussion preceding Proposition 1.8).

**Definition 1.1.2 (Direct Factors of Free pro- $p$  Groups).** Let  $F$  be a free pro- $p$  group, and  $H$  a closed subgroup of  $F$ . Let  $\iota : H \rightarrow F$  be the natural homomorphism. We say that  $H$  is a direct factor of  $F$  if there exists a continuous homomorphism  $s : F \rightarrow H$  such that  $s \circ \iota = \text{id}_H$  [ $s$  is necessarily surjective]. We then have a natural exact sequence

$$1 \rightarrow N \rightarrow F \xrightarrow{s} H \rightarrow 1,$$

where  $N \stackrel{\text{def}}{=} \text{Ker } s$ , and  $F$  is isomorphic to the free direct product

$$H * N$$

In particular,  $N$  is also a direct factor of  $F$  (cf. loc. cit).

One has the following cohomological characterization of direct factors of free pro- $p$  groups.

**Proposition 1.1.3.** *Let  $H$  be a pro- $p$  group, and  $F$  a free pro- $p$  group. Let  $\sigma : H \rightarrow F$  be a continuous homomorphism. Assume that the map induced by  $\sigma$  on cohomology:*

$$h^1(\sigma) : H^1(F, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^1(H, \mathbb{Z}/p\mathbb{Z})$$

*is surjective. [Here  $\mathbb{Z}/p\mathbb{Z}$  is considered as a trivial discrete module]. Then  $H$  is a direct factor of  $F$ .*

*Proof.* cf. [Ga], Proposition 1.8.  $\square$

**1.2 Formal Patching.** In this sub-section we explain the procedure which allows to construct [Galois] covers of curves in the setting of formal geometry, by patching together covers of formal affine curves with covers of formal fibres at closed points of the special fibre (cf. [Sa], 1, for more details). We also recall the [well-known] local-global principle for liftings of Galois covers of curves.

Let  $R$  be a complete discrete valuation ring, with fraction field  $K$ , residue field  $k$ , and uniformiser  $\pi$ . Let  $X$  be an admissible formal  $R$ -scheme which is an  $R$ -curve, by which we mean that the special fibre  $X_k \stackrel{\text{def}}{=} X \times_R k$  is a reduced one-dimensional scheme of finite type over  $k$ . Let  $Z$  be a finite set of closed points of  $X$ . For a point  $x \in Z$ , let  $X_x \stackrel{\text{def}}{=} \text{Spf } \hat{\mathcal{O}}_{X,x}$  be the formal completion of  $X$  at  $x$ , which is the formal fibre at the point  $x$ . Let  $X'$  be a formal open sub-scheme of  $X$  whose special fibre is  $X_k \setminus Z$ .

For each closed point  $x \in Z$ , let  $\{\mathcal{P}_i\}_{i=1}^n$  be the set of minimal prime ideals of  $\hat{\mathcal{O}}_{X,x}$  which contain  $\pi$ ; they correspond to the branches  $\{\eta_i\}_{i=1}^n$  of the completion of  $X_k$  at  $x$ , and let  $X_{x,i} \stackrel{\text{def}}{=} \text{Spf } \hat{\mathcal{O}}_{x,\mathcal{P}_i}$  be the formal completion of the localisation of  $X_x$  at  $\mathcal{P}_i$ . The local ring  $\hat{\mathcal{O}}_{x,\mathcal{P}_i}$  is a complete discrete valuation ring. The set  $\{X_{x,i}\}_{i=1}^n$  is the set of boundaries of the formal fibre  $X_x$ . For each  $i \in \{1, \dots, n\}$ , we have a canonical morphism  $X_{x,i} \rightarrow X_x$ .

**Definition 1.2.1.** With the same notations as above, a  $(G)$ -cover patching data for the pair  $(X, Z)$  consists of the following.

- (i) A finite (Galois) cover  $Y' \rightarrow X'$  (with Galois group  $G$ ).
- (ii) For each point  $x \in Z$ , a finite (Galois) cover  $Y_x \rightarrow X_x$  (with Galois group  $G$ ).

The above data (i) and (ii) must satisfy the following compatibility condition.

(iii) If  $\{X_{x,i}\}_{i=1}^n$  are the boundaries of the formal fibre at the point  $x$ , then for each  $i \in \{1, \dots, n\}$  is given a ( $G$ -equivariant)  $X_x$ -isomorphism

$$\sigma_i : Y_x \times_{X_x} X_{x,i} \xrightarrow{\sim} Y' \times_{X'} X_{x,i}.$$

Property (iii) should hold for each  $x \in Z$ .

The following is the main patching result that we will use in this paper.

**Proposition 1.2.2.** *With the same notations as above. Given a ( $G$ -)cover patching data as in Definition 1.2.1, there exists a unique, up to isomorphism, ( $G$ -)cover  $Y \rightarrow X$  (with Galois group  $G$ ) which induces the above ( $G$ -)cover in Definition 1.2.1, (i), when restricted to  $X'$ , and induces the above ( $G$ -)cover in Definition 1.2.1, (ii), when pulled-back to  $X_x$ , for each point  $x \in Z$ .*

**1.2.3.** With the same notations as above, let  $x \in Z$ , and  $\tilde{X}_k$  the normalisation of  $X_k$ . There is a one-to-one correspondence between the set of points of  $\tilde{X}_k$  above  $x$ , and the set of boundaries of the formal fibre at the point  $x$ . Let  $x_i$  be the point of  $\tilde{X}_k$  above  $x$  which corresponds to the boundary  $X_{x,i}$ , for  $i \in \{1, \dots, n\}$ . Assume that the point  $x \in X_k(k)$  is rational. Then the completion of  $\tilde{X}_k$  at  $x_i$  is isomorphic to the spectrum of a ring of formal power series  $k[[t_i]]$  in one variable over  $k$ , where  $t_i$  is a local parameter at  $x_i$ .

The complete local ring  $\hat{\mathcal{O}}_{x,\mathcal{P}_i}$  is a discrete valuation ring with residue field isomorphic to  $k((t_i))$ . Let  $T_i \in \hat{\mathcal{O}}_{x,\mathcal{P}_i}$  be an element which lifts  $t_i$ . Such an element is called a parameter of  $\hat{\mathcal{O}}_{x,\mathcal{P}_i}$ . Then there exists an isomorphism  $\hat{\mathcal{O}}_{x,\mathcal{P}_i} \xrightarrow{\sim} R[[T_i]]\{T_i^{-1}\}$ , where

$$R[[T]]\{T^{-1}\} \stackrel{\text{def}}{=} \left\{ \sum_{i=-\infty}^{\infty} a_i T^i, \lim_{i \rightarrow -\infty} |a_i| = 0 \right\},$$

and  $|\cdot|$  is a normalised absolute value of  $R$ .

As a direct consequence of the above patching result, and the theorems of liftings of étale covers (cf. [Gr]), one obtains the following [well-known] local-global principle for liftings of ( $G$ -)covers of curves.

**Proposition 1.2.4.** *Let  $X$  be a proper, flat, algebraic (or formal)  $R$ -curve, and let  $Z \stackrel{\text{def}}{=} \{x_i\}_{i=1}^n$  be a finite set of closed points of  $X$ . Let  $f_k : Y_k \rightarrow X_k$  be a finite generically separable ( $G$ -)cover (with Galois group  $G$ ), whose branch locus is contained in  $Z$ . Assume that for each  $i \in \{1, \dots, n\}$ , there exists a ( $G$ -)cover  $f_i : Y_i \rightarrow \text{Spf } \hat{\mathcal{O}}_{X,x_i}$  (with Galois group  $G$ ) which lifts the cover  $\hat{Y}_{k,x_i} \rightarrow \text{Spec } \hat{\mathcal{O}}_{X_k,x_i}$  induced by  $f_k$ , where  $\hat{\mathcal{O}}_{X_k,x_i}$  (resp.  $\hat{Y}_{k,x_i}$ ) denotes the completion of  $X_k$  at  $x_i$  (resp. the completion of  $Y_k$  above  $x_i$ ). Then there exists a unique, up to isomorphism, ( $G$ -)cover  $f : Y \rightarrow X$  (with Galois group  $G$ ) which lifts the cover  $f_k$ , and which is isomorphic to the cover  $f_i$  when pulled back to  $\text{Spf } \hat{\mathcal{O}}_{X,x_i}$ , for each  $i \in \{1, \dots, n\}$ .*

**1.3 Degeneration of  $\mu_p$ -torsors.** In this sub-section, we recall the [well-known] degeneration of  $\mu_p$ -torsors from zero to positive characteristics, above the boundaries of formal fibres of formal  $R$ -curves at closed points.

Here  $R$  denotes a complete discrete valuation ring of unequal characteristics, with fraction field  $K$ , residue field  $k$  of characteristic  $p > 0$ , uniformiser  $\pi$ , and



which contains  $\zeta$ : a primitive  $p$ -th root of 1. We write  $\lambda \stackrel{\text{def}}{=} \zeta - 1$ . We denote by  $v_K$  the valuation of  $K$ , which is normalised by  $v_K(\pi) = 1$ .

First, we recall the definition of a certain class of  $R$ -group schemes (cf. [Se-Oo-Su], for more details).

**1.3.1 Torsors under finite and flat  $R$ -group schemes of rank  $p$ : the group schemes  $\mathcal{G}_n$  and  $\mathcal{H}_n$ .** Let  $n \geq 1$  be an integer. Define the affine  $R$ -group scheme

$$\mathcal{G}_{n,R} \stackrel{\text{def}}{=} \text{Spec}(A_n)$$

as follows.

- (i)  $A_n \stackrel{\text{def}}{=} R[X, \frac{1}{1+\pi^n X}]$ .
- (ii) The comultiplication  $c_n : A_n \rightarrow A_n \otimes_R A_n$  is defined by  $c_n(X) \stackrel{\text{def}}{=} X \otimes 1 + 1 \otimes X + \pi^n X \otimes X$ .
- (iii) The coinverse  $i_n : A_n \rightarrow A_n$  is defined by  $i_n(X) \stackrel{\text{def}}{=} -\frac{X}{1+\pi^n X}$ .
- (iv) The counit  $\epsilon_n : A_n \rightarrow R$  is defined by  $\epsilon_n(X) \stackrel{\text{def}}{=} 0$ .

One verifies [easily] that  $\mathcal{G}_n \stackrel{\text{def}}{=} \mathcal{G}_{n,R}$  is an affine, commutative, and smooth  $R$ -group scheme, with generic fibre  $(\mathcal{G}_n)_K \xrightarrow{\sim} \mathbb{G}_{m,K}$ , and special fibre  $(\mathcal{G}_n)_k \xrightarrow{\sim} \mathbb{G}_{a,k}$ .

Next, we introduce some finite and flat group schemes of rank  $p$ . Assume that  $n$  satisfies the following condition

$$(*) \quad 0 < n(p-1) \leq v_K(p).$$

Assuming that the above condition  $(*)$  holds, consider the map

$$\phi_n : \mathcal{G}_n \rightarrow \mathcal{G}_{pn},$$

given by

$$X \mapsto \frac{(1 + \pi^n X)^p - 1}{\pi^{pn}}.$$

Then  $\phi_n$  is a surjective homomorphism of  $R$ -group schemes. Denote by

$$\mathcal{H}_n \stackrel{\text{def}}{=} \mathcal{H}_{n,R} \stackrel{\text{def}}{=} \text{Ker}(\phi_n).$$

It is a finite and flat commutative group scheme of rank  $p$ . Under the assumption  $(*)$  one verifies [easily] that the generic fibre  $\mathcal{H}_{n,K} \stackrel{\text{def}}{=} \mathcal{H}_n \otimes_R K \xrightarrow{\sim} \mu_{p,K}$  is étale, and the special fibre  $\mathcal{H}_{n,k} \stackrel{\text{def}}{=} \mathcal{H}_n \otimes_R k \xrightarrow{\sim} \alpha_{p,k}$  is radicial of type  $\alpha_p$ , if  $n(p-1) < v_K(\lambda)$ , and  $\mathcal{H}_{n,k} \stackrel{\text{def}}{=} \mathcal{H}_n \otimes_R k \xrightarrow{\sim} (\mathbb{Z}/p\mathbb{Z})_k$  is étale, if  $n(p-1) = v_K(\lambda)$ .

Let  $\mathcal{U} \stackrel{\text{def}}{=} \text{Spf } A$  be a formal affine  $R$ -scheme, and  $f : \mathcal{V} \rightarrow \mathcal{U}$  a torsor under the finite group scheme  $\mathcal{H}_n$ , for some  $n$  as above satisfying  $(*)$ . Then there exists a regular function  $u \in A$ , such that the image  $\bar{u}$  of  $u$  in  $\bar{A} \stackrel{\text{def}}{=} A/\pi A$  is not a  $p$  power if  $n(p-1) < v_K(\lambda)$ ,  $1 + \pi^{pn}u$  is defined up to multiplication by a  $p$ -th power of the form  $(1 + \pi^n v)^p$ , and the torsor  $f$  is given by an equation  $(X')^p = (1 + \pi^n X)^p = 1 + \pi^{pn}u$ , where  $X'$  and  $X$  are indeterminates. Moreover, the natural morphism  $f_k : \mathcal{V}_k \rightarrow \mathcal{U}_k$  between the special fibres is either the  $\alpha_p$ -torsor given by the equation  $x^p = \bar{u}$ , where  $x = X \bmod \pi$ , and  $\bar{u} = u \bmod \pi$ , if  $n(p-1) < v_K(\lambda)$ . Or, is the  $\mathbb{Z}/p\mathbb{Z}$ -torsor

given by the equation  $x^p - x = \bar{u}$  where  $x = X \bmod \pi$ , and  $\bar{u} = u \bmod \pi$ , if  $n(p-1) = v_K(\lambda)$

Next, we recall the degeneration of  $\mu_p$ -torsors on the boundary  $\mathcal{X} \stackrel{\text{def}}{=} \text{Spf } R[[T]]\{T^{-1}\}$  of formal fibres of germs of formal  $R$ -curves. Here  $R[[T]]\{T^{-1}\}$  is as in 1.2.3. Note that  $R[[T]]\{T^{-1}\}$  is a complete discrete valuation ring, with uniformising parameter  $\pi$ , and residue field  $k((t))$ , where  $t = T \bmod \pi$ . The following result will be used in §3 and §4.

**Proposition 1.3.2.** *Let  $A \stackrel{\text{def}}{=} R[[T]]\{T^{-1}\}$  (cf. above definition), and  $f : \text{Spf } B \rightarrow \text{Spf } A$  a non-trivial Galois cover of degree  $p$ . Assume that the ramification index of the corresponding extension of discrete valuation rings equals 1. Then  $f$  is a torsor under a finite and flat  $R$ -group scheme  $G$  of rank  $p$ . Let  $\delta$  be the degree of the different in the above extension. The following cases occur.*

(a)  $\delta = v_K(p)$ . Then  $f$  is a torsor under the group scheme  $G = \mu_{p,R}$ , and two cases occur.

(a1) For a suitable choice of the parameter  $T$  of  $A$  the torsor  $f$  is given, after eventually a finite extension of  $R$ , by an equation  $Z^p = T^h$ . In this case we say that the torsor  $f$  has a degeneration of type  $(\mu_p, 0, h)$ .

(a2) For a suitable choice of the parameter  $T$  of  $A$  the torsor  $f$  is given, after eventually a finite extension of  $R$ , by an equation  $Z^p = 1 + T^m$  where  $m$  is a positive integer prime to  $p$ . In this case we say that the torsor  $f$  has a degeneration of type  $(\mu_p, -m, 0)$ .

(b)  $0 < \delta < v_K(p)$ . Then  $f$  is a torsor under the group scheme  $\mathcal{H}_{n,R}$ , where  $n$  is such that  $\delta = v_K(p) - n(p-1)$ . Moreover, for a suitable choice of the parameter  $T$  the torsor  $f$  is given, after eventually a finite extension of  $R$ , by an equation  $Z^p = 1 + \pi^{pn}T^m$ , with  $m \in \mathbb{Z}$  prime to  $p$ . In this case we say that the torsor  $f$  has a degeneration of type  $(\alpha_p, -m, 0)$ .

(c)  $\delta = 0$ . Then  $f$  is an étale torsor under the  $R$ -group scheme  $G = \mathcal{H}_{v_K(\lambda),R}$ , and is given, after eventually a finite extension of  $R$ , by an equation  $Z^p = \lambda^p T^m + 1$ , where  $m$  is a negative integer prime to  $p$ , for a suitable choice of the parameter  $T$  of  $A$ . In this case we say that the torsor  $f$  has a degeneration of type  $(\mathbb{Z}/p\mathbb{Z}, -m, 0)$ .

*Proof.* See [Sa], Proposition 2.3.  $\square$

## §2. Pro- $p$ Quotients of the Geometric Galois Group of a $p$ -adic Open Disc.

**2.1 Notations.** The following notations will be used in this section and the subsequent ones, unless we specify otherwise.

$p > 1$  is a fixed prime integer.

$R$  will denote a complete discrete valuation ring of unequal characteristics, with uniformising parameter  $\pi$ .

$K \stackrel{\text{def}}{=} \text{Fr}(R)$  is the quotient field of  $R$ ,  $\text{char}(K) = 0$ .

$k \stackrel{\text{def}}{=} R/\pi R$  is the residue field of  $R$ , which we assume to be algebraically closed of characteristic  $p > 0$ .

$v_K$  will denote the valuation of  $K$ , which is normalised by  $v_K(\pi) = 1$ .

For an  $R$ -(formal) scheme  $X$  we will denote by  $X_K \stackrel{\text{def}}{=} X \times_R K$  (resp.  $X_k \stackrel{\text{def}}{=} X \times_R k$ ) the generic (resp. special) fibre of  $X$ .

**2.2.** Next, we would like to state a result of Garuti, in [Ga], which concerns the structure of the pro- $p$  geometric fundamental group of a  $p$ -adic annulus of thickness zero (cf. Proposition 2.2.3). First, we recall how one defines the fundamental group of a rigid analytic affinoid space.

Let  $\mathcal{X} = \mathrm{Spf} \mathcal{A}$  be an affine  $R$ -formal scheme which is topologically of finite type. Thus,  $\mathcal{A}$  is a  $\pi$ -adically complete noetherian  $R$ -algebra. Let  $A \stackrel{\mathrm{def}}{=} \mathcal{A} \otimes_R K$  be the corresponding Tate algebra, and  $X \stackrel{\mathrm{def}}{=} \mathrm{Sp} A$  the associated rigid analytic affinoid space, which is the generic fiber of  $\mathcal{X}$  in the sense of Raynaud (cf. [Ab]).

Assume that  $X$  is integral and geometrically connected. Let  $\eta$  be a geometric point of the affine scheme  $\mathrm{Spec} A$  above the generic point of  $\mathrm{Spec} A$ . Then  $\eta$  determines naturally an algebraic closure  $\overline{K}$  of  $K$ , and a geometric point of  $\mathrm{Spec}(A \times_K \overline{K})$ , which we will also denote by  $\eta$ .

**Definition 2.2.1 (Étale Fundamental Groups of Affinoid Spaces).** (See also [Ga], Définition 2.2, and Définition 2.3). We define the étale fundamental group of  $X$  with base point  $\eta$  by

$$\pi_1(X, \eta) \stackrel{\mathrm{def}}{=} \pi_1(\mathrm{Spec} A, \eta),$$

where  $\pi_1(\mathrm{Spec} A, \eta)$  is the étale fundamental group of the connected scheme  $\mathrm{Spec} A$  with base point  $\eta$  in the sense of Grothendieck (cf. [Gr]). Thus,  $\pi_1(X, \eta)$  naturally classifies rigid analytic coverings  $Y \rightarrow X$ , where  $Y = \mathrm{Sp} B$ , and  $B$  is a finite  $A$ -algebra which is étale over  $A$ .

There exists a natural continuous surjective homomorphism

$$\pi_1(X, \eta) \twoheadrightarrow \mathrm{Gal}(\overline{K}, K).$$

We define the geometric fundamental group  $\pi_1(X, \eta)^{\mathrm{geo}}$  of  $X$  so that the following sequence is exact:

$$1 \rightarrow \pi_1(X, \eta)^{\mathrm{geo}} \rightarrow \pi_1(X, \eta) \rightarrow \mathrm{Gal}(\overline{K}, K) \rightarrow 1.$$

**Remark 2.2.2.** If  $L/K$  is a finite field extension contained in  $\overline{K}/K$ , and  $X_L \stackrel{\mathrm{def}}{=} X \times_K L$  is the affinoid rigid analytic space obtained from  $X$  by extending scalars, then we have a natural commutative diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(X_L, \eta)^{\mathrm{geo}} & \longrightarrow & \pi_1(X_L, \eta) & \longrightarrow & \mathrm{Gal}(\overline{K}/L) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \pi_1(X, \eta)^{\mathrm{geo}} & \longrightarrow & \pi_1(X, \eta) & \longrightarrow & \mathrm{Gal}(\overline{K}/K) \longrightarrow 1 \end{array}$$

where the two right vertical maps are injective homomorphisms, and the left vertical map is an isomorphism.

The geometric fundamental group  $\pi_1(X, \eta)^{\mathrm{geo}}$  is strictly speaking not the fundamental group of a rigid analytic space [since  $\overline{K}$  is not complete]. It is, however, the projective limit of fundamental groups of rigid affinoid spaces. More precisely, there exists a natural isomorphism

$$\pi_1(X, \eta)^{\mathrm{geo}} \xrightarrow{\sim} \varprojlim_{L/K} \pi_1(X \times_K L, \eta),$$

where the limit is taken over all finite extensions  $L/K$  contained in  $\overline{K}$ .

Next, we introduce some notations involved in the statement of Garuti's result. For a finite field extension  $L/K$  contained in  $\overline{K}/K$  we will denote by

$$D_0 \stackrel{\text{def}}{=} D_{0,L} \stackrel{\text{def}}{=} \text{Sp } L < X >$$

the unit closed disc [centred at  $X = 0$ ], and

$$C_L \stackrel{\text{def}}{=} \text{Sp } \frac{L < X, Y >}{(XY - 1)}$$

the annulus of thickness 0 which is the “boundary” of  $D_0$ . Here  $L < X >$  (resp.  $L < X, Y >$ ) denotes the Tate algebra in the variable  $X$  (resp. the variables  $X$  and  $Y$ ).

We denote by  $\mathbb{P}_L^1$  the rigid analytic projective line over  $L$  which is obtained by patching the closed discs  $D_0 = D_{0,L} \stackrel{\text{def}}{=} \text{Sp } L < X >$ , and  $D_\infty = D_{\infty,L} \stackrel{\text{def}}{=} \text{Sp } L < Y >$ , along the annulus  $C_L$  (see above), via the identification  $X \mapsto \frac{1}{Y}$ .

Let  $S = \{a_1, a_2, \dots, a_n\}$  be a finite set of closed points of  $\mathbb{P}_K^1$  which contains  $\{0, \infty\}$ , and such that  $S \cap C_K = \emptyset$ . We view  $S \subset \mathbb{P}_K^1$  as a closed subscheme of  $\mathbb{P}_K^1$ , and write  $S_L \stackrel{\text{def}}{=} S \times_K L$ . Let  $\eta$  be a geometric point of  $\mathbb{P}_{\overline{K}}^1 \stackrel{\text{def}}{=} \mathbb{P}_K^1 \times_K \overline{K}$  above the generic point of  $\mathbb{P}_{\overline{K}}^1$ . We denote by  $\pi_1(\mathbb{P}_L^1 \setminus S_L, \eta)$  the algebraic étale fundamental group of  $\mathbb{P}_L^1 \setminus S_L$  with base point  $\eta$ . Write

$$C \stackrel{\text{def}}{=} C_K \stackrel{\text{def}}{=} \text{Sp } \frac{K < X, Y >}{(XY - 1)}.$$

The natural embedding  $C \times_K L = C_L \rightarrow \mathbb{P}_L^1$  induces a natural continuous homomorphism

$$\pi_1(C, \eta)^{\text{geo}} \rightarrow \pi_1(\mathbb{P}_L^1 \setminus S_L, \eta),$$

and by passing to the projective limit a continuous homomorphism

$$\pi_1(C, \eta)^{\text{geo}} \rightarrow \pi_1(\mathbb{P}_{\overline{K}}^1 \setminus S_{\overline{K}}, \eta) \stackrel{\text{def}}{=} \varprojlim_{L/K} \pi_1(\mathbb{P}_L^1 \setminus S_L, \eta),$$

where  $L/K$  runs over all finite extensions contained in  $\overline{K}$ .

Let  $\pi_1(C, \eta)^{\text{geo}, p}$  be the maximal pro- $p$  quotient of  $\pi_1(C, \eta)^{\text{geo}}$ , and  $\pi_1(\mathbb{P}_{\overline{K}}^1 \setminus S_{\overline{K}}, \eta)^p$  the maximal pro- $p$  quotient of  $\pi_1(\mathbb{P}_{\overline{K}}^1 \setminus S_{\overline{K}}, \eta)$ . The above homomorphism  $\pi_1(C, \eta)^{\text{geo}} \rightarrow \pi_1(\mathbb{P}_{\overline{K}}^1 \setminus S_{\overline{K}}, \eta)$  induces a natural continuous homomorphism

$$\phi_S : \pi_1(C, \eta)^{\text{geo}, p} \rightarrow \pi_1(\mathbb{P}_{\overline{K}}^1 \setminus S_{\overline{K}}, \eta)^p,$$

which induces, by passing to the projective limit, a continuous homomorphism

$$\phi \stackrel{\text{def}}{=} \varprojlim_S \phi_S : \pi_1(C, \eta)^{\text{geo}, p} \rightarrow \varprojlim_S \pi_1(\mathbb{P}_{\overline{K}}^1 \setminus S_{\overline{K}}, \eta)^p,$$

where the limit is taken over all finite set of closed points of  $\mathbb{P}_{\overline{K}}^1 \setminus C$  which contain  $\{0, \infty\}$ .

The profinite pro- $p$  group  $\varprojlim_S \pi_1(\mathbb{P}_{\overline{K}}^1 \setminus S_{\overline{K}}, \eta)^p$  is a free pro- $p$  group, as follows from the well-known structure of algebraic fundamental groups in characteristic 0 (cf. [Gr]).

The following result is one of the main technical results in [Ga], which we will use in this section (cf. proof of Theorem 2.3.1, and proof of Theorem 2.4.1).

**Proposition 2.2.3 (Garuti).** *The natural continuous homomorphism*

$$\phi \stackrel{\text{def}}{=} \varprojlim_S \phi_S : \pi_1(C, \eta)^{\text{geo}, p} \rightarrow \varprojlim_S \pi_1(\mathbb{P}_K^1 \setminus S_{\overline{K}}, \eta)^p,$$

where the limit is taken over all finite set of closed points of  $\mathbb{P}_K^1 \setminus C$  which contain  $\{0, \infty\}$ , makes  $\pi_1(C, \eta)^{\text{geo}, p}$  into a direct factor of  $\varprojlim_S \pi_1(\mathbb{P}_K^1 \setminus S, \eta)^p$ . In particular,  $\pi_1(C, \eta)^{\text{geo}, p}$  is a free pro- $p$  group.

*Proof.* See [Ga], Lemma 2.11.  $\square$

**2.3.** Next, we will investigate the structure of a certain quotient of the “geometric absolute Galois group” of a  $p$ -adic open disc. First, we will define this quotient (see the definition of the profinite group  $\Delta$  below).

Write

$$\tilde{X} \stackrel{\text{def}}{=} \text{Spec } R[[T]],$$

and

$$\tilde{X}_K \stackrel{\text{def}}{=} \tilde{X} \times_R K = \text{Spec}(R[[T]] \otimes_R K).$$

[ $\tilde{X}_K$  is what we shall refer to as a  $p$ -adic open disc (over  $K$ )].

Let  $\tilde{S} \stackrel{\text{def}}{=} \{x_1, x_2, \dots, x_n\} \subset \tilde{X}_K$  be a finite set of closed points of  $\tilde{X}_K$ . We view  $\tilde{S} \subset \tilde{X}_K$  as a closed subscheme of  $\tilde{X}_K$ . Write

$$U_{K, \tilde{S}} \stackrel{\text{def}}{=} \tilde{X}_K \setminus \tilde{S},$$

and let  $\eta$  be a geometric point of  $\tilde{X}_K$  above the generic point of  $\tilde{X}_K$ . We have a natural exact sequence of profinite groups:

$$1 \rightarrow \pi_1(U_{K, \tilde{S}} \times_K \overline{K}, \eta) \rightarrow \pi_1(U_{K, \tilde{S}}, \eta) \rightarrow \text{Gal}(\overline{K}, K) \rightarrow 1.$$

By passing to the projective limit over all finite set of closed points  $\tilde{S} \subset \tilde{X}_K$ , we obtain a natural exact sequence:

$$1 \rightarrow \varprojlim_{\tilde{S}} \pi_1(U_{K, \tilde{S}} \times_K \overline{K}, \eta) \rightarrow \varprojlim_{\tilde{S}} \pi_1(U_{K, \tilde{S}}, \eta) \rightarrow \text{Gal}(\overline{K}/K) \rightarrow 1.$$

Let  $L \stackrel{\text{def}}{=} \text{Fr}(R[[T]])$  be the quotient field of the formal power series ring  $R[[T]]$ . The generic point  $\eta$  determines an algebraic closure  $\overline{L}$  of  $L$ . We have a natural exact sequence of Galois groups:

$$1 \rightarrow \text{Gal}(\overline{L}/L.\overline{K}) \rightarrow \text{Gal}(\overline{L}/L) \rightarrow \text{Gal}(\overline{K}/K) \rightarrow 1.$$

Moreover, there exist natural identifications:

$$\text{Gal}(\overline{L}/\overline{K}.L) \xrightarrow{\sim} \varprojlim_{\tilde{S}} \pi_1(U_{K, \tilde{S}} \times_K \overline{K}, \eta),$$

and

$$\text{Gal}(\overline{L}/L) \xrightarrow{\sim} \varprojlim_{\tilde{S}} \pi_1(U_{K, \tilde{S}}, \eta),$$

where  $\tilde{S}$  is as above  $[\text{Gal}(\overline{L}/\overline{K}.L)$  is what we shall refer to as the geometric Galois group of a  $p$ -adic open disc].

Let

$$I \stackrel{\text{def}}{=} I_{(\pi)} \subset \text{Gal}(\overline{L}.\overline{K}.L)$$

be the normal closed subgroup which is generated by the inertia subgroups above the ideal  $(\pi)$  of  $R[[T]]$ , which is generated by  $\pi$ . Write

$$\overline{\Delta} \stackrel{\text{def}}{=} \text{Gal}(\overline{L}/\overline{K}.L)/I.$$

Note that, by definition, the profinite group  $\overline{\Delta}$  classifies finite Galois covers  $\tilde{Y}_{K'} \rightarrow \tilde{X}_{K'} \stackrel{\text{def}}{=} \tilde{X} \times_R K'$ , where  $K'$  is a finite extension of  $K$  with valuation ring  $R'$ ,  $K'$  is algebraically closed in  $\tilde{Y}_{K'}$ ,  $\pi'$  is a uniformising parameter of  $K'$ , and the natural morphism  $\tilde{Y}' \rightarrow \tilde{X}' \stackrel{\text{def}}{=} \tilde{X} \times_R R'$ , where  $\tilde{Y}'$  is the normalisation of  $\tilde{X}'$  in  $\tilde{Y}_{K'}$ , is étale above the generic point of the special fiber  $\tilde{X}'_k \stackrel{\text{def}}{=} \tilde{X}' \times_{R'} k$  of  $\tilde{X}'$ . In particular, the special fiber  $\tilde{Y}'_k \stackrel{\text{def}}{=} \tilde{Y}' \times_{R'} k$  is reduced and the natural morphism  $\tilde{Y}'_k \rightarrow \tilde{X}'_k$  is generically étale.

Let

$$\Delta \stackrel{\text{def}}{=} \overline{\Delta}^p$$

be the maximal pro- $p$  quotient of  $\overline{\Delta}$ . Our main technical result in this section is the following:

**Theorem 2.3.1.** *The profinite group  $\Delta$  is a free pro- $p$  group.*

*Proof.* The two main ingredients of the proof are the technical result of Garuti in Proposition 2.2.3, and a result of Harbater, Katz, and Gabber (cf. [Ha], and [Ka]).

We will show that the profinite pro- $p$  group  $\Delta$  satisfies property (iii) in Proposition 1.1.1. Let  $Q \twoheadrightarrow P$  be a surjective homomorphism between finite  $p$ -groups. Let  $\phi : \Delta \twoheadrightarrow P$  be a surjective homomorphism. We will show that  $\phi$  lifts to a homomorphism  $\psi : \Delta \rightarrow Q$ . The homomorphism  $\phi$  corresponds to a finite Galois extension  $\overline{L}'/\overline{K}.L$  with Galois group  $P$ . We can [without loss of generality] assume that this extension is defined over  $K$ , thus descends to a finite Galois extension  $L'/L$  with Galois group  $P$  where  $K$  is algebraically closed in  $L'$ .

Let  $A$  be the integral closure of  $R[[T]]$  in  $L'$ . We have a finite morphism  $f : \text{Spec } A \rightarrow \text{Spec } R[[T]]$  which is [by assumptions] Galois with Galois group  $P$ , is étale above the point  $(\pi) \in \text{Spec } R[[T]]$ , and with  $\text{Spec } A$  geometrically connected. In particular,  $f$  induces at the level of special fibres a finite generically Galois cover  $\bar{f} : \text{Spec}(A/\pi A) \rightarrow \text{Spec } k[[t]]$  [where  $t = T \bmod \pi$ ] with Galois group  $P$ . We will assume [in order to simplify the arguments below] that  $\text{Spec}(A/\pi A)$  is connected, the general case is treated in a similar fashion.

By a result of Harbater, Katz and Gabber (cf. [Ha], and [Ka]) there exists a finite Galois cover  $\bar{g} : \overline{Y} \rightarrow \mathbb{P}_k^1$  with Galois group  $P$ , which is étale outside a unique closed point  $\infty$  of  $\mathbb{P}_k^1$  with local parameter  $t$ ,  $\bar{g}$  is totally ramified above  $\infty$ ,  $\overline{Y}$  is connected, and such that the Galois cover above the formal completion  $\text{Spec } \hat{\mathcal{O}}_{\mathbb{P}_k^1, \infty}$  of  $\mathbb{P}_k^1$  at  $\infty$ , which is naturally induced by  $\bar{g}$ , is isomorphic to  $\bar{f}$ . Let  $\mathbb{A}_k^1 \stackrel{\text{def}}{=} \mathbb{P}_k^1 \setminus \{\infty\}$ . The restriction  $\bar{f}' : \overline{Y}' \rightarrow \mathbb{A}_k^1$  of  $\bar{g}$  to  $\mathbb{A}_k^1$  is an étale Galois cover with Galois group  $P$ , and  $\overline{Y}'$  is connected.

Consider the rigid analytic projective line  $\mathbb{P}_K^1$  which is obtained by patching the closed unit disc  $D_\infty \stackrel{\text{def}}{=} \text{Sp } K < T > [\text{centered at } T = \infty]$  with the closed disc  $D_0 \stackrel{\text{def}}{=} \text{Sp } K < S > [\text{centered at } S = 0]$  along the annulus  $C \stackrel{\text{def}}{=} \text{Sp } K < T, S > / (ST - 1)$  [of thickness 0], via the identification  $S \mapsto \frac{1}{T}$ . The étale Galois cover  $\bar{f}'$  lifts [uniquely up to isomorphism] to an étale Galois cover  $f' : Y' \rightarrow D_0$  by the theorems of liftings of étale covers (cf. [Gr]), whose restriction  $\tilde{f}' : \tilde{Y}' \rightarrow C$  to the annulus  $C$  is an étale Galois cover with Galois group  $P$ .

By using formal patching techniques à la Harbater one can construct a [connected] rigid analytic Galois cover  $g : Y \rightarrow \mathbb{P}_K^1$  with Galois group  $P$  whose restriction to the annulus  $C$  is isomorphic to  $\tilde{f}'$ , and which above the formal completion at  $T = \infty$  induces the above Galois cover  $f$ . (see the arguments used in [Ga], and Proposition 1.2.2).

The Galois cover  $g$  is ramified above a finite set of closed points  $\tilde{S} \subset \mathbb{P}_K^1$  [which are contained in the interior  $D_\infty^\circ$  of the closed disc  $D_\infty$ ], hence gives rise naturally to a surjective homomorphism  $\phi_1 : \pi_1(\mathbb{P}_K^1 \setminus \tilde{S}_{\overline{K}}, \eta) \twoheadrightarrow P$ , and also a surjective homomorphism  $\varprojlim_S \pi_1(\mathbb{P}_K^1 \setminus S_{\overline{K}}, \eta)^p \twoheadrightarrow P$  [where  $S$  and the projective limit are as in Proposition 2.2.3]. We also denote by  $\phi_1 : \pi_1(C, \eta)^{\text{geo}, p} \twoheadrightarrow P$  the corresponding homomorphism induced on the direct factor  $\pi_1(C, \eta)^{\text{geo}, p}$  of  $\varprojlim_S \pi_1(\mathbb{P}_K^1 \setminus S_{\overline{K}}, \eta)^p$ .

Next, let  $\bar{f}_1 : \text{Spec } \overline{B} \rightarrow \text{Spec } k[[t]]$  be a finite connected Galois cover with Galois group  $Q$  which dominates the above Galois cover  $\bar{f} : \text{Spec}(A/\pi A) \rightarrow \text{Spec } k[[t]]$  with Galois group  $P$ . Note that such  $\bar{f}_1$  exists, since the maximal pro- $p$  quotient of the absolute Galois group of  $k((t))$  is a free pro- $p$  group. Let  $\bar{g}_1 : \overline{Y}_1 \rightarrow \mathbb{P}_k^1$  be the finite Galois cover with Galois group  $Q$  which is étale outside  $\infty$ , which induces above  $\text{Spec } \hat{\mathcal{O}}_{\mathbb{P}_k^1, \infty}$  a finite Galois cover which is isomorphic to  $\bar{f}_1$ , and let  $\bar{g}'_1 : \overline{Y}'_1 \rightarrow \mathbb{A}_k^1$  be its restriction to  $\mathbb{A}_k^1$  [the Galois cover  $\bar{g}_1$  exists by the above result of Harbater, Katz and Gabber (cf. loc. cit.)]. By construction the Galois cover  $\bar{g}_1 : \overline{Y}_1 \rightarrow \mathbb{P}_k^1$  dominates the Galois cover  $\bar{g} : \overline{Y} \rightarrow \mathbb{P}_k^1$ . The étale Galois cover  $\bar{g}'_1$  lifts to a finite étale Galois cover  $f'_1 : Y'_1 \rightarrow D_0$  with Galois group  $Q$ , which by construction dominates the lifting  $f' : Y' \rightarrow D_0$  of  $\bar{f}'$ . The restriction of  $f'_1$  to the annulus  $C$  is a finite étale Galois cover  $\tilde{f}'_1 : \tilde{Y}'_1 \rightarrow C$  with Galois group  $Q$  which dominates the Galois cover  $\tilde{f}' : \tilde{Y}' \rightarrow C$ .

Let  $N$  be a complement of  $\pi_1(C, \eta)^{\text{geo}, p}$  in  $\varprojlim_S \pi_1(\mathbb{P}_K^1 \setminus S_{\overline{K}}, \eta)^p$  (cf. Proposition 2.2.3). The Galois cover  $\tilde{f}'_1 : \tilde{Y}'_1 \rightarrow C$  (resp.  $\tilde{f}' : \tilde{Y}' \rightarrow C$ ) corresponds to the continuous homomorphism  $\phi_2 : \pi_1(C, \eta)^{\text{geo}} \rightarrow Q$  (resp.  $\phi_1 : \pi_1(C, \eta)^{\text{geo}, p} \rightarrow P$ ), and  $\phi_2$  dominates  $\phi_1$  [by construction]. Also the above homomorphism  $\phi_1 : \varprojlim_S \pi_1(\mathbb{P}_K^1 \setminus S_{\overline{K}}, \eta)^p \twoheadrightarrow P$  induces naturally a continuous homomorphism  $\psi_1 : N \rightarrow P$ .

The pro- $p$  group  $N$  being free one can lift the homomorphism  $\psi_1$  to a homomorphism  $\psi_2 : N \rightarrow Q$  which dominates  $\psi_1$ . The profinite pro- $p$  group  $\varprojlim_S \pi_1(\mathbb{P}_K^1 \setminus S_{\overline{K}}, \eta)^p$  being the free direct product of  $N$  and  $\pi_1(C, \eta)^{\text{geo}, p}$  one can construct a continuous homomorphism  $\psi : \varprojlim_S \pi_1(\mathbb{P}_K^1 \setminus S_{\overline{K}}, \eta)^p \rightarrow Q$  which restricts to  $\phi_2$  on the factor  $\pi_1(C, \eta)^{\text{geo}, p}$ , and to  $\psi_2$  on the factor  $N$ . Moreover,  $\psi : \pi_1(\mathbb{P}_K^1 \setminus (S_0)_{\overline{K}}, \eta)^p \rightarrow Q$  factors through  $\psi : \pi_1(\mathbb{P}_K^1 \setminus (S_0)_{\overline{K}}, \eta)^p$  for some set of closed points  $S_0 \subset \mathbb{P}_K^1$ , and  $S_0 \cap C = \emptyset$ . The homomorphism  $\psi$  corresponds [after eventually a finite extension of  $K$ ] to a Galois cover  $Y \rightarrow \mathbb{P}_K^1$  with Galois group  $Q$ , which induces naturally a

finite Galois cover  $g : \text{Spec } B \rightarrow \text{Spec } R[[T]]$  with Galois group  $Q$  (above the formal completion at  $T = \infty$ ), which is by construction étale above the ideal  $\pi$ , and which dominates the Galois cover  $f : \text{Spec } A \rightarrow \text{Spec } R[[T]]$  we started with. This in turn corresponds to a homomorphism  $\psi : \Delta \rightarrow Q$  with the required properties.  $\square$

The author doesn't know, and is interested to know, the answer to the following question.

**Questions 2.3.2.** Is the maximal pro- $p$  quotient  $\text{Gal}(\overline{L}/\overline{K}.L)^p$  of the [geometric] Galois group  $\text{Gal}(\overline{L}/\overline{K}.L)$  a free pro- $p$  group?

**2.4.** Next, we investigate a certain quotient of the "geometric absolute Galois group" of the boundary of a  $p$ -adic open disc (see the definition of the profinite groups  $\Delta'$ , and  $\Pi'$  below).

Let  $R[[T]]\{T^{-1}\} \stackrel{\text{def}}{=} \{\sum_{i=-\infty}^{\infty} a_i T^i, \lim_{i \rightarrow -\infty} |a_i| = 0\}$  be as in 1.2.3. Note that  $R[[T]]\{T^{-1}\}$  is a complete discrete valuation ring, with uniformising parameter  $\pi$ , and residue field the formal power series field  $k((t))$ , where  $t = T \bmod \pi$ . Write

$$\mathcal{X} \stackrel{\text{def}}{=} \text{Spec } R[[T]]\{T^{-1}\}.$$

$[\mathcal{X}$  is what we shall refer to as the boundary of a  $p$ -adic open disc (over  $K$ )]. Let

$$M \stackrel{\text{def}}{=} \text{Fr}(R[[T]]\{T^{-1}\})$$

be the quotient field of the discrete valuation ring  $R[[T]]\{T^{-1}\}$ .

Assume that the generic point  $\eta$  of  $\tilde{X}_K$  above arises from a generic point  $\eta$  of  $R[[T]]\{T^{-1}\} \otimes_R K$ . In particular, the generic point  $\eta$  determines then an algebraic closure  $\overline{M}$  of  $M$ . We have a natural exact sequence of Galois groups

$$1 \rightarrow \text{Gal}(\overline{M}/M.\overline{K}) \rightarrow \text{Gal}(\overline{M}/M) \rightarrow \text{Gal}(\overline{K}/K) \rightarrow 1.$$

Let  $I' \stackrel{\text{def}}{=} I'_{(\pi)} \subset \text{Gal}(\overline{M}/\overline{K}.M)$  be the normal closed subgroup which is generated by the inertia subgroups above the ideal  $(\pi)$  of  $R[[T]]\{T^{-1}\}$ , which is generated by  $\pi$ . Write

$$\overline{\Delta}' \stackrel{\text{def}}{=} \text{Gal}(\overline{M}/\overline{K}.M)/I'.$$

Note that, by definition, the profinite group  $\overline{\Delta}'$  classifies finite Galois covers  $\mathcal{Y}_L \rightarrow \mathcal{X}_L \stackrel{\text{def}}{=} \mathcal{X} \times_R L$ , where  $L$  is a finite extension of  $K$  with valuation ring  $R'$ ,  $L$  is algebraically closed in  $\mathcal{Y}_L$ ,  $\pi'$  is a uniformising parameter of  $L$ , and the natural morphism  $\mathcal{Y} \rightarrow \mathcal{X}' \stackrel{\text{def}}{=} \mathcal{X} \times_R R'$  where  $\mathcal{Y}$  is the normalisation of  $\mathcal{X}'$  in  $\mathcal{Y}_L$  is étale.

The natural morphism

$$\text{Spec } R[[T]]\{T^{-1}\} \rightarrow \text{Spec } R[[T]]$$

induces a natural homomorphism

$$\overline{\Delta}' \rightarrow \overline{\Delta}.$$

Let

$$\Delta' \stackrel{\text{def}}{=} \overline{\Delta}'^p$$

be the maximal pro- $p$  quotient of  $\overline{\Delta}'$ . We have a natural homomorphism

$$\Delta' \rightarrow \Delta.$$

Our next technical result in this section is the following.



**Theorem 2.4.1.** *There exists a natural homomorphism  $\Delta' \rightarrow \Delta$  which makes  $\Delta'$  into a direct factor of the free pro- $p$  group  $\Delta$ . In particular,  $\Delta'$  is a free pro- $p$  group [this can also be deduced from the fact that the maximal pro- $p$  quotient of the absolute Galois group of the field  $k((t))$  is a free pro- $p$  group].*

*Proof.* One has to verify the cohomological criterion in Proposition 1.1.3 for being a direct factor.

Let  $f' : \mathcal{Y} \rightarrow \mathcal{X}$  be an étale  $\mathbb{Z}/p\mathbb{Z}$ -torsor. One has to construct [eventually after a finite extension of  $K$ ] a finite generically Galois cover  $f : \tilde{Y} \rightarrow \tilde{X} \stackrel{\text{def}}{=} \text{Spec } R[[T]]$  of degree  $p$  which induces above  $\mathcal{X}$ , by pull-back via the natural morphism  $\mathcal{X} \stackrel{\text{def}}{=} \text{Spec } R[[T]]\{T^{-1}\} \rightarrow \tilde{X} \stackrel{\text{def}}{=} \text{Spec } R[[T]]$ , the  $\mathbb{Z}/p\mathbb{Z}$ -torsor  $f'$ .

The torsor  $f'$  induces naturally a finite generically Galois cover  $\bar{f}' : \mathcal{Y}_k \rightarrow \mathcal{X}_k = \text{Spec } k[[t]]$  of degree  $p$ . There exists [as is easily verified (cf. also [Ka])] a finite Galois cover  $\bar{g} : Y_k \rightarrow \mathbb{P}_k^1$  of degree  $p$  which is ramified above a unique point  $\infty \in \mathbb{P}_k^1$ , and such that the Galois cover induced by  $\bar{g}$  above the formal completion  $\text{Spec } \hat{\mathcal{O}}_{\mathbb{P}_k^1, \infty}$  of  $\mathbb{P}_k^1$  at  $\infty$  is isomorphic to  $\bar{f}'$ . Let  $\bar{g}' : Y'_k \rightarrow \mathbb{A}_k^1 \stackrel{\text{def}}{=} \mathbb{P}_k^1 \setminus \{\infty\}$  be the restriction of  $\bar{g}$  which is an étale  $\mathbb{Z}/p\mathbb{Z}$ -torsor above  $\mathbb{A}_k^1$ . The étale torsor  $\bar{g}'$  lifts [uniquely up to isomorphism] to an étale  $\mathbb{Z}/p\mathbb{Z}$ -torsor  $g' : Y' \rightarrow D_0 \stackrel{\text{def}}{=} \text{Sp } K < S >$  [where  $D_0$  is the closed disc centered at  $S = 0$ ], by the theorems of liftings of étale covers (cf. [Gr]), whose restriction  $\tilde{g}' : \tilde{Y}' \rightarrow C$  to the annulus  $C$  is an étale  $\mathbb{Z}/p\mathbb{Z}$ -torsor, which corresponds to a continuous homomorphism  $\psi : \pi_1(C, \eta)^{\text{geo}, p} \twoheadrightarrow \mathbb{Z}/p\mathbb{Z}$ .

The geometric fundamental group  $\pi_1(C, \eta)^{\text{geo}, p}$  being a direct factor of  $\varprojlim_S \pi_1(\mathbb{P}_{\bar{K}}^1 \setminus S_{\bar{K}}, \eta)^p$ , the above homomorphism  $\psi$  arises [by restriction] from a continuous homomorphism  $\psi' : \varprojlim_S \pi_1(\mathbb{P}_{\bar{K}}^1 \setminus S_{\bar{K}}, \eta)^p \twoheadrightarrow \mathbb{Z}/p\mathbb{Z}$ , and the later gives rise naturally to a Galois cover  $g : Y \rightarrow \mathbb{P}_{\bar{K}}^1$  of degree  $p$  [this cover only exists a priori over a finite extension of  $K$  but we can, without loss of generality, assume that it is defined over  $K$ ] whose restriction to the annulus  $C$  is isomorphic [by construction] to the above Galois cover  $\tilde{g}'$ . The Galois cover  $g$  induces naturally a Galois cover above  $\tilde{X} \stackrel{\text{def}}{=} \text{Spec } R[[T]]$  (i.e. above the formal completion at  $T = \infty$ ), which induces above the boundary  $\mathcal{X}$  the torsor  $f'$  as required.  $\square$

**2.5.** In [Ga], Garuti investigated the problem of lifting of Galois covers between smooth curves. In this sub-section we will prove a refined version of the main result in [Ga], using Theorem 2.3.1 and Theorem 2.4.1.

First, we recall the following main result of Garuti.

**Theorem 2.5.1 (Garuti).** *Let  $X$  be a proper, smooth, and geometrically connected  $R$ -curve. Let*

$$f_k : Y_k \rightarrow X_k \stackrel{\text{def}}{=} X \times_R k$$

*be a finite [possibly ramified] Galois cover between smooth  $k$ -curves with Galois group  $G$ . Then there exists a finite extension  $R'/R$  and a finite morphism*

$$f' : Y' \rightarrow X' \stackrel{\text{def}}{=} X \times_R R',$$

*which is generically étale and Galois with Galois group  $G$ , satisfying the following properties:*

(i)  $Y'$  is a proper and normal  $R'$ -curve.

(ii) The natural morphism  $f'_k : Y'_k \stackrel{\text{def}}{=} Y' \times_R k \rightarrow X'_k = X_k$  is generically étale and Galois with Galois group  $G$ . Moreover, there exists a  $G$ -equivariant birational morphism  $\nu : Y_k \rightarrow Y'_k \stackrel{\text{def}}{=} Y' \times_R k$  such that the following diagram is commutative:

$$\begin{array}{ccc} Y_k & \xrightarrow{\nu} & Y'_k \\ f_k \downarrow & & f'_k \downarrow \\ X_k & \xrightarrow{\text{id}_{X_k}} & X_k \end{array}$$

and the morphism  $\nu$  is an isomorphism outside the divisor of ramification in the morphism  $f_k : Y_k \rightarrow X_k$ .

(iii) The special fiber  $Y'_k$  is reduced, unibranche, and the morphism  $\nu : Y_k \rightarrow Y'_k$  is a morphism of normalisation. In particular,  $Y_k$  and  $Y'_k$  are homeomorphic.

*Proof.* (cf. [Ga], Proof of Théorème 2).  $\square$

In light of the above result, we define Garuti liftings as follows.

**Definition 2.5.2 (Garuti Liftings of Galois Covers between Smooth Curves).** Let  $X$  be a proper, smooth, and geometrically connected  $R$ -curve. Let

$$f_k : Y_k \rightarrow X_k \stackrel{\text{def}}{=} X \times_R k$$

be a finite [possibly ramified] Galois cover with Galois group  $G$ . Let

$$f' : Y' \rightarrow X' \stackrel{\text{def}}{=} X \times_R R'$$

be as in Theorem 2.5.1, for some finite extension  $R'/R$ . We call  $f'$  a Garuti lifting of the Galois cover  $f_k$  [defined over  $R'$ ].

We say that  $f'$  is a smooth lifting of  $f_k$ , if  $Y'$  is a smooth  $R'$ -curve, which is equivalent to the above morphism  $\nu : Y_k \rightarrow Y'_k$  being an isomorphism.

Note that, by definition, a Garuti lifting is defined [a priori] over a finite extension of  $R$ . Also, if  $f_k$  is étale, then a smooth lifting of  $f_k$  always exists over  $R$  as follows from the theorems of liftings of étale covers (cf. [Gr]).

The following Theorem is a refined version of the above result of Garuti.

**Theorem 2.5.3.** *Let  $X$  be a proper, smooth, and geometrically connected  $R$ -curve. Let*

$$f_k : Y_k \rightarrow X_k \stackrel{\text{def}}{=} X \times_R k$$

*be a finite [possibly ramified] Galois cover with Galois group  $G$  between smooth  $k$ -curves. Assume that the finite group  $G$  sits in an exact sequence*

$$1 \rightarrow H' \rightarrow G \rightarrow H \rightarrow 1.$$

*Let*

$$Y_k \xrightarrow{g_k} Z_k \xrightarrow{h_k} X_k$$

be the corresponding factorisation of the Galois cover  $f_k$ . Thus,  $h_k : Z_k \rightarrow X_k$  is a finite Galois cover with Galois group  $H$  between smooth  $k$ -curves. Let

$$h' : Z' \rightarrow X' \stackrel{\text{def}}{=} X \times_R R'$$

be a Garuti lifting of the Galois cover  $h_k$  defined over the finite extension  $R'/R$  (cf. Definition 2.5.2).

Then there exists a finite extension  $R''/R'$ , and a Garuti lifting

$$f'' : Y'' \rightarrow X'' \stackrel{\text{def}}{=} X \times_R R''$$

of the Galois cover  $f_k$  over  $R''$ , which dominates  $h'$ , i.e. we have a factorisation

$$f'' : Y'' \xrightarrow{g''} Z'' \stackrel{\text{def}}{=} Z \times_{R'} R'' \xrightarrow{h'' \stackrel{\text{def}}{=} h' \times_{R'} R''} X'',$$

where  $g'' : Y'' \rightarrow Z''$  is a finite morphism between normal  $R''$ -curves.

*Proof.* The proof is [in some sense] similar to the proof of Théorème 2 in [Ga] using the above Theorem 2.4.1. More precisely, using the techniques of formal patching (cf. [Ga], and Proposition 1.2.2) the proof of Theorem 2.5.3 follows directly from the following local result in Theorem 2.5.5.  $\square$

Before stating our main local result, we first define the local analog of Garuti liftings.

**Definition 2.5.4 (Local Garuti Liftings).** Let  $\tilde{X} \stackrel{\text{def}}{=} \text{Spec } R[[T]]$ , and  $\tilde{X}_k \stackrel{\text{def}}{=} \text{Spec } k[[t]]$ . Let  $G$  be a finite group and

$$f_k : \tilde{Y}_k \rightarrow \tilde{X}_k$$

a finite morphism, which is generically Galois with Galois group  $G$ , with  $\tilde{Y}_k$  connected and normal. We call a Garuti lifting of the Galois cover  $f_k$ , over the finite extension  $R'/R$ , a finite Galois cover

$$f' : \tilde{Y}' \rightarrow \tilde{X}' \stackrel{\text{def}}{=} X \times_R R'$$

with Galois group  $G$ , where  $R'/R$  is a finite extension, the morphism  $f'_k : \tilde{Y}'_k \rightarrow \tilde{X}_k$  is generically Galois with Galois group  $G$ , there exists a birational  $G$ -equivariant morphism  $\nu : \tilde{Y}_k \rightarrow \tilde{Y}'_k$  which is a morphism of normalisation, and a factorisation

$$f_k : \tilde{Y}_k \xrightarrow{\nu} \tilde{Y}'_k \xrightarrow{f'_k} \tilde{X}_k.$$

Moreover, we say that  $f$  is a smooth lifting of  $f_k$  if  $\tilde{Y}'$  is smooth over  $R'$ , or equivalently if the above morphism  $\nu$  is an isomorphism.

The following is our main result which is a refined version of the local version of Garuti's main Theorem 2.5.1.

**Theorem 2.5.5.** *Let  $\tilde{X} \stackrel{\text{def}}{=} \text{Spec } R[[T]]$ , and  $\tilde{X}_k \stackrel{\text{def}}{=} \text{Spec } k[[t]]$ . Let  $G$  be a finite group and*

$$f_k : \tilde{Y}_k \rightarrow \tilde{X}_k$$

*a finite morphism which is generically Galois with Galois group  $G$ , with  $\tilde{Y}_k$  normal and connected. Let  $H$  be a quotient of  $G$  and*

$$h_k : \tilde{Z}_k \rightarrow \tilde{X}_k$$

*the corresponding Galois sub-cover with Galois group  $H$ . Let*

$$h' : \tilde{Z}' \rightarrow \tilde{X}' \stackrel{\text{def}}{=} \tilde{X} \times_R R'$$

*be a Garuti lifting of  $h_k$  over a finite extension  $R'/R$  (cf. Definition 2.5.4). Then there exists a finite extension  $R''/R'$ , and a Garuti lifting*

$$f'' : \tilde{Y}'' \rightarrow \tilde{X}'' \stackrel{\text{def}}{=} \tilde{X} \times_R R''$$

*of  $f_k$  over  $R''$  which dominates  $h'$ , i.e. we have a factorisation:*

$$f'' : \tilde{Y}'' \rightarrow \tilde{Z}'' \stackrel{\text{def}}{=} \tilde{Z}' \times_{R'} R'' \xrightarrow{h'' \stackrel{\text{def}}{=} h' \times_{R'} R''} \tilde{X}''.$$

*Proof.* The Galois group  $G$  is a solvable group which is a semi-direct product of a cyclic group of order prime to  $p$  by a  $p$ -group. By similar arguments as the ones used by Garuti in [Ga], it suffices to treat the case where  $G$  is a  $p$ -group (see the arguments used in [Ga], Théorème 2.13, and Corollaire 1.11). In this case the proof follows from Theorem 2.4.1.

More precisely, assume that  $G$  is a  $p$ -group [hence  $H$  is also a  $p$ -group]. The Galois cover  $f_k : \tilde{Y}_k \rightarrow \tilde{X}_k$  is generically given by an étale Galois cover  $\text{Spec } k((s)) \rightarrow \text{Spec } k((t))$  with Galois group  $G$ . This étale cover lifts uniquely to an étale Galois cover  $\mathcal{Y} \rightarrow \mathcal{X} \stackrel{\text{def}}{=} \text{Spec } R[[T]]\{T^{-1}\}$  above the boundary of the open disc  $\tilde{X}$ , which is Galois with Galois group  $G$ , and which corresponds to a continuous homomorphism  $\psi_2 : \Delta' \rightarrow G$ .

Let  $N$  be a complement of  $\Delta'$  in  $\Delta$  (cf. Theorem 2.4.1). The local Garuti lifting  $h' : \tilde{Z}' \rightarrow \tilde{X}' \stackrel{\text{def}}{=} \tilde{X} \times_R R'$  corresponds to a continuous homomorphism  $\phi : \Delta \rightarrow H$ , which restricts to continuous homomorphisms  $\psi_1 : \Delta' \rightarrow H$ , and  $\phi_1 : N \rightarrow H$ . The above homomorphism  $\psi_2$  dominates by construction the homomorphism  $\psi_1$ . The pro- $p$  group  $N$  being free one can lift the homomorphism  $\phi_1$  to a continuous homomorphism  $\phi_2 : N \rightarrow G$  which dominates  $\phi_1$ . The pro- $p$  group  $\Delta$  being isomorphic to the direct free product  $\Delta' \star N$ , both  $\psi_2$  and  $\phi_2$  give rise to a continuous homomorphism  $\phi' : \Delta \rightarrow G$  which dominates the above morphism  $\phi$ . The homomorphism  $\phi'$  in turn corresponds to a Galois cover  $\tilde{Y}'' \rightarrow \tilde{X}'' \stackrel{\text{def}}{=} \tilde{X} \times_R R''$  over some finite extension  $R''/R$ , which is a Garuti lifting of  $f_k : \tilde{Y}_k \rightarrow \tilde{X}_k$ , and which by construction dominates the Garuti lifting  $h' : \tilde{Z}' \rightarrow \tilde{X}' \stackrel{\text{def}}{=} \tilde{X} \times_R R'$  of the sub-cover  $h_k : \tilde{Z}_k \rightarrow \tilde{X}_k$  as required.  $\square$

**Remark 2.5.6.** We assumed in this section that  $R$  is of unequal characteristics. In fact the main results of this section: Theorem 2.3.1, Theorem 2.4.1, Theorem 2.5.3, and Theorem 2.5.5, are also valid in the case of a complete discrete valuation ring  $R$  of equal characteristics  $p > 0$ . Indeed, the result of Garuti (cf. Proposition 2.2.3) that we use in the proof of Theorem 2.3.1, and Theorem 2.4.1, is valid in this case (cf. [Ga]).

**§3. Fake Liftings of Cyclic Covers between Smooth Curves.** In this section we use the same notations as in §2, 2.1. We will investigate the problem of lifting of cyclic [of  $p$ -power order] Galois covers between smooth curves.

**3.1 The Oort Conjecture.** First, we recall the following main conjecture which was formulated by F. Oort, and several of its variants. In what follows  $R$  is as in the Notations 2.1.

**The Original Oort Conjecture [Conj-O].** (cf. [Oo], and [Oo1]) Let

$$f_k : Y_k \rightarrow X_k$$

be a finite [possibly ramified] Galois cover between smooth  $k$ -curves, with Galois group  $G \xrightarrow{\sim} \mathbb{Z}/m\mathbb{Z}$  a cyclic group. Then there exists a finite extension  $R'/R$ , and a Galois cover

$$f : Y' \rightarrow X'$$

with Galois group  $G$ , where  $X'$  and  $Y'$  are smooth  $R'$ -curves, which lifts the Galois cover  $f_k$ , i.e. the morphism induced by  $f$  at the level of special fibres is [Galois] isomorphic to  $f_k$ .

In the original version of the conjecture, one doesn't fix  $R$ , but fixes  $k$ ,  $f_k$ , and asks for the existence of a local domain  $R$  dominating the ring of Witt vectors  $W(k)$ , over which a lifting of  $f_k$  exists, as part of the conjecture (cf. [Oo]).

One can formulate several variants of the above conjecture, that we will list below.

**[Conj-O1].** Let  $X$  be a proper, smooth, geometrically connected  $R$ -curve, and  $f_k : Y_k \rightarrow X_k \stackrel{\text{def}}{=} X \times_R k$  a finite [possibly ramified] Galois cover between smooth  $k$ -curves, with Galois group  $G \xrightarrow{\sim} \mathbb{Z}/m\mathbb{Z}$  a cyclic group. Then there exists a smooth lifting of  $f_k$  (cf. Definition 2.5.2), i.e. there exists a finite extension  $R'/R$ , and a Galois cover  $f' : Y' \rightarrow X' \stackrel{\text{def}}{=} X \times_R R'$  between smooth  $R'$ -curves, with Galois group  $G$ , such that the special fiber  $X'_k \stackrel{\text{def}}{=} X' \times_{R'} k$  (resp.  $Y'_k \stackrel{\text{def}}{=} Y' \times_{R'} k$ ) equals  $X_k$  (resp. is isomorphic to  $Y_k$ ), and the natural morphism  $f'_k \stackrel{\text{def}}{=} f' \times_{R'} k : Y'_k \rightarrow X'_k = X_k$  which is induced by  $f'$  on the level of special fibres is isomorphic to  $f_k$ .

**[Conj-O2].** Let  $f_k : Y_k \rightarrow \mathbb{P}_k^1$  be a finite ramified Galois cover, with  $Y_k$  a smooth  $k$ -curve, and Galois group  $G \xrightarrow{\sim} \mathbb{Z}/m\mathbb{Z}$  a cyclic group. Then there exists a smooth lifting of  $f_k$  (cf. Definition 2.5.2), i.e. there exists a finite extension  $R'/R$ , a finite Galois cover  $f' : Y' \rightarrow \mathbb{P}_{R'}^1$ , with  $Y'$  a smooth  $R'$ -curve, with Galois group  $G$ , and such that the natural morphism  $f'_k \stackrel{\text{def}}{=} f' \times_R k : Y'_k \rightarrow \mathbb{P}_k^1$  which is induced by  $f'$  on the level of special fibres is isomorphic to  $f_k$ .

**[Conj-O3].** Let  $f_k : Y_k \rightarrow \mathbb{P}_k^1$  be a finite Galois cover, with  $Y_k$  a smooth  $k$ -curve, and Galois group  $G \xrightarrow{\sim} \mathbb{Z}/m\mathbb{Z}$  a cyclic group, which is [totally] ramified above a unique point  $\infty \in \mathbb{P}_k^1$ . Then there exists a smooth lifting of  $f_k$  (cf. Definition 2.5.2), i.e. there exists a finite extension  $R'/R$ , a finite Galois cover  $f' : Y' \rightarrow \mathbb{P}_{R'}^1$ , with  $Y'$  a smooth  $R'$ -curve, with Galois group  $G$ , and such that the natural morphism  $f'_k \stackrel{\text{def}}{=} f' \times_R k : Y'_k \rightarrow \mathbb{P}_k^1$  which is induced by  $f'$  on the level of special fibres is isomorphic to  $f_k$ .

**[Conj-O4].** Let  $\tilde{X} \stackrel{\text{def}}{=} \text{Spec } R[[T]]$ , and  $\tilde{X}_k \stackrel{\text{def}}{=} \text{Spec } k[[t]]$ . Let  $f_k : \tilde{Y}_k \rightarrow \tilde{X}_k$  be a finite morphism which is generically Galois with Galois group  $G \xrightarrow{\sim} \mathbb{Z}/m\mathbb{Z}$  a cyclic group, with  $\tilde{Y}_k$  normal and connected. Then there exists a finite extension  $R'/R$ , and a smooth lifting  $f' : \tilde{Y}' \rightarrow \tilde{X}' \stackrel{\text{def}}{=} \tilde{X} \times_R R'$  of  $f_k$ , i.e.  $\tilde{Y}' \xrightarrow{\sim} \text{Spec } R'[[T]]$  is  $R'$ -smooth, and the natural morphism  $f'_k : \tilde{Y}'_k \rightarrow \tilde{X}'_k = \tilde{X}_k$  which is induced by  $f'$  at the level of special fibres is isomorphic to  $f_k$ .

Moreover, in the above conjectures **[Conj-O1]**, **[Conj-O2]**, **[Conj-O3]**, and **[Conj-O4]**, one predicts that  $R'$  can be chosen to be the minimal extension of  $R$  which contains a primitive  $m$ -th root of 1.

In fact all the above variants of the Oort conjecture turn out to be equivalent. More precisely, we have the following.

**Lemma 3.1.1.** *With the above notations, the various conjectures **[Conj-O]**, **[Conj-O1]**, **[Conj-O2]**, **[Conj-O3]**, and **[Conj-O4]**, are all equivalent. Moreover, in order to solve the above conjecture(s), it suffices to treat the case where  $G \xrightarrow{\sim} \mathbb{Z}/p^n\mathbb{Z}$  is a cyclic  $p$ -group.*

*Proof.* Follows easily from the local-global principle for the lifting of Galois covers between curves (cf. Proposition 1.2.4), the result of approximation of local extensions by global extensions due to Katz, Gabber, and Harbater, (cf. [Ha], and [Ka]), and the formal patching result in Proposition 1.2.2. The last assertion can also be easily verified (see for example the arguments in [Gr-Ma], 6).  $\square$

Oort conjecture holds true in the case where the Galois cover  $f_k$  is étale, as follows from the theorems of liftings of étale covers (cf. [Gr]). In this case the statement of the conjecture is true for any finite group  $G$  [not necessarily cyclic], and a smooth lifting exists over  $R$ . In the case where  $G \xrightarrow{\sim} \mathbb{Z}/p^n\mathbb{Z}$  is a cyclic  $p$ -group, the conjecture has been verified in the cases where  $n = 1$ , and  $n = 2$  (cf. [Se-Oo-Su] for the case  $n = 1$ , and [Gr-Ma] for the case  $n = 2$ ).

In this paper, and in light of Theorem 2.5.3, we propose the following refined version of the Oort conjecture. More precisely, we will formulate a refined version of **[Conj-O1]**, which is equivalent to **[Conj-O]** by Lemma 3.1.1.

**Oort Conjecture Revisited [Conj-O1-Rev].** Let  $X$  be a proper, smooth, geometrically connected  $R$ -curve, and  $f_k : Y_k \rightarrow X_k \stackrel{\text{def}}{=} X \times_R k$  a finite [possibly ramified] Galois cover between smooth  $k$ -curves, with Galois group  $G \xrightarrow{\sim} \mathbb{Z}/m\mathbb{Z}$  a cyclic group. Let  $H$  be a quotient of  $G$ , and  $g_k : Z_k \rightarrow X_k$  the corresponding Galois sub-cover of  $f_k$  with Galois group  $H$ . Then there exists a smooth Galois lifting

$$g : Z' \rightarrow X' \stackrel{\text{def}}{=} X \times_R R'$$

of  $g_k$ , over some finite extension  $R'/R$  [i.e.  $g$  is a Galois cover with Galois group  $H$  between smooth  $R'$ -curves which is a lifting of  $g_k$ ].

Furthermore, for every smooth lifting  $g$  of the Galois sub-cover  $g_k$  of  $f_k$  as above, there exists a finite extension  $R''/R'$ , and a finite Galois cover

$$f : Y'' \rightarrow X'' \stackrel{\text{def}}{=} X \times_R R''$$

between smooth  $R''$ -curves, with Galois group  $G$ , which is a smooth lifting of  $f_k$  (cf. Definition 2.5.2), and such that  $f$  dominates  $g$ , i.e. we have a factorisation

$$f : Y'' \rightarrow Z'' \stackrel{\text{def}}{=} Z' \times_{R'} R'' \xrightarrow{g \times_{R'} R''} X''.$$

Moreover,  $R''$  can be chosen to be the minimal extension of  $R'$  which contains a primitive  $m$ -th root of 1.

**Remark 3.1.2.** In a similar way, one can revisit the above [equivalent] variants of the original Oort conjecture, and formulate the revisited versions [**Conj-O2-Rev**], [**Conj-O3-Rev**], and [**Conj-O4-Rev**], which turn out to be all equivalent to [**Conj-O1-Rev**] (use similar arguments as in the proof of Lemma 3.1.1). Moreover, and in order to solve these revisited versions, one can reduce to the case where  $G \xrightarrow{\sim} \mathbb{Z}/p^n\mathbb{Z}$  is a cyclic  $p$ -group. In the case where  $n = 1$  [i.e.  $G$  is a cyclic group of cardinality  $p$ ] the revisited Oort conjecture is clearly true, since the [original] Oort conjecture is true in this case (see [Se-Oo-Su]). Both the original and the revisited conjectures are clearly equivalent in this case.

**3.2.** Next, we give examples where the revisited Oort conjecture can be verified in the case where  $G \xrightarrow{\sim} \mathbb{Z}/p^2\mathbb{Z}$ .

We assume that  $G \xrightarrow{\sim} \mathbb{Z}/p^2\mathbb{Z}$  is cyclic of order  $p^2$ . We will work within the framework of [**Conj-O4-Rev**] (cf. Remark 3.1.2). More precisely, let  $\tilde{X} \stackrel{\text{def}}{=} \text{Spec } R[[T]]$ , and  $\tilde{X}_k \stackrel{\text{def}}{=} \text{Spec } k[[t]]$  its special fiber [where  $t = T \bmod \pi$ ]. Let

$$f_k : Y_k \rightarrow \tilde{X}_k$$

be a cyclic Galois cover of degree  $p^2$ , with  $Y_k$  normal, and

$$h_k : Y'_k \rightarrow \tilde{X}_k$$

its unique Galois sub-cover of degree  $p$ . A smooth local lifting of  $f_k$  [cf. Definition 2.5.4] exists by [Gr-Ma], Theorem 5.5, over  $R$  if  $R$  contains the  $p^2$ -th roots of 1. From now on we will assume in this sub-section that  $R$  contains a primitive  $p^2$ -th root of 1. Let

$$h : Y' \rightarrow \tilde{X}$$

be a smooth Galois lifting of  $h_k$ , i.e.  $h$  is a Galois cover of degree  $p$ ,  $Y' \stackrel{\text{def}}{=} \text{Spec } A'$ ,  $A' \xrightarrow{\sim} R[[Z']]$  is an open disc, and  $h$  induces the Galois cover  $h_k$  on the level of special fibres. Then, in order to verify the [**Conj-O4-Rev**] for the Galois cover  $f_k$  and the smooth lifting  $h$  of  $h_k$ , it suffices to show that there exists a smooth Galois lifting

$$f : Y \rightarrow \tilde{X}$$

of  $f_k$ , i.e.  $f$  is a cyclic Galois cover of degree  $p^2$ ,  $Y \stackrel{\text{def}}{=} \text{Spec } A$ ,  $A \xrightarrow{\sim} R[[Z]]$  is an open disc, and  $f$  induces the Galois cover  $f_k$  on the level of special fibres, which dominates  $h$ : i.e. such that we have a factorisation

$$f : Y \rightarrow Y' \xrightarrow{h} \tilde{X}.$$

The Galois cover  $f_k$  is generically given, for an appropriate choice of the parameter  $t$ , by the equations:

$$(*) \quad x_1^p - x_1 = t^{-m_1},$$

and

$$(**) \quad x_2^p - x_2 = c(x_1^p, -x_1) + \sum_{0 \leq s < m_1(p-1)} a_s t^{-s} + \sum_{0 \leq j < m_1} t^{-jp} \sum_{0 < i < p} (x_1^p - x_1)^i p_{j,p-i} (x_1^p - x_1)^p,$$

where  $a_i \in k$ ,  $p_{j,p-i} \in k[x]$  are polynomials of respective degrees  $d_{j,p-i}$ ,  $\gcd(m_1, p) = 1$ , and

$$c(x, y) \stackrel{\text{def}}{=} \frac{(x+y)^p - x^p + (-y^p)}{p}.$$

(See [Gr-Ma], Lemma 5.1). Moreover, the degree of the different in the Galois cover  $f_k$  is

$$d_s \stackrel{\text{def}}{=} (m_1 + 1)(p-1)p + (m_2 + 1)(p-1),$$

where

$$m_2 \stackrel{\text{def}}{=} \max_{\substack{0 \leq j < m_1 \\ 0 < i < p}} (p^2 m_1, p(jp + (i + p d_{j,p-i}) m_1)) - (p-1)m_1,$$

(cf. loc. cit).

Let  $\zeta_2 \in R$  be a primitive  $p^2$ -th root of 1. Let  $\zeta_1 \stackrel{\text{def}}{=} \zeta_2^p$ , and  $\lambda \stackrel{\text{def}}{=} \zeta_1 - 1$ . The smooth lifting  $h : Y' \rightarrow \tilde{X}$  of  $h_k$  is generically given [by the Oort-Sekiguchi-Suwa theory (cf. [Se-Oo-Su])] by an equation

$$\frac{(\lambda X_1 + 1)^p - 1}{\lambda^p} = f(T),$$

where

$$f(T) = \frac{h(T)}{g(T)},$$

$h(T) \in R[[T]]$ ,  $g(T) \in R[T]$  is a distinguished polynomial [i.e. its highest coefficient is a unit in  $R$ ], the degree of  $g(T)$  is  $m$ , the Weierstrass degree of  $h(T)$  is  $m'$ ,  $m \geq m'$ , and  $m - m' = m_1$ . Furthermore,

$$\frac{h(T)}{g(T)} = T^{-m_1} \mod \pi.$$

The smoothness of  $Y'$  is equivalent, by the local criterion for smoothness (cf. [Gr-Ma], 3.4), to the fact that the Galois cover  $h_K : Y'_K \rightarrow \tilde{X}_K$  which is induced by  $h$  between generic fibres, and which is given by the equation

$$(\lambda X_1 + 1)^p = \frac{\lambda^p h(T) + g(T)}{g(T)},$$



is ramified above  $m_1 + 1$  distinct geometric points of  $\tilde{X}_K$ . Moreover,  $Z' \stackrel{\text{def}}{=} X_1^{-\frac{1}{m_1}}$  is a parameter for the open disc  $Y'$ , as follows easily from arguments similar to the one given in the proof of Theorem 4.1 in [Gr-Ma] (cf. also [Gr-Ma], proof of 3.4).

We will consider two cases, depending on the lift  $h$  of  $h_k$ , where we can prove the revisited Oort conjecture [**Conj-O4-Rev**] for the smooth lifting  $h : Y' \rightarrow \tilde{X}$  [i.e. we can dominate  $h$  by a smooth lifting  $f$  of  $f_k$ ]. These two cases are considered separately in the following lemmas 3.2.1 and 3.2.2.

**Lemma 3.2.1.** *With the same notations as above. Assume that in the second equation (\*\*) above defining the Galois cover  $f_k$  we have*

$$\sum_{0 \leq s < m_1(p-1)} a_s t^{-s} + \sum_{0 \leq j < m_1} t^{-jp} \sum_{0 < i < p} (x_1^p - x_1)^i p_{j,p-i} (x_1^p - x_1)^p = 0,$$

and also assume that the degree of  $g(T)$  above equals  $m_1$ . [In particular,  $h(T) \in R[[T]]$  above is a unit in this case]. Then there exists a smooth lifting  $f$  of  $f_k$  which dominates the smooth lifting  $h$  of  $h_k$ . In particular, [**Conj-O4-rev**] is true under these conditions for the Galois cover  $f_k$ , and the smooth lifting  $h$  of the sub-cover  $h_k$ .

*Proof.* Consider the cover

$$f : Y \rightarrow \tilde{X}$$

which is generically given by the equations

$$(i) \quad \frac{(\lambda X_1 + 1)^p - 1}{\lambda^p} = f(T),$$

where  $f(T) = h(T)/g(T)$  is as above, and

$$(ii) \quad (\lambda X_2 + \text{Exp}_p(\mu X_1))^p = (\lambda X_1 + 1) \text{Exp}_p(\mu^p Y),$$

where

$$\text{Exp}_p X \stackrel{\text{def}}{=} 1 + X + \dots + \frac{X^{p-1}}{(p-1)!}$$

is the truncated exponential,

$$\mu \stackrel{\text{def}}{=} \log_p(\zeta_2) = 1 - \zeta_2 + \dots + (-1)^{p-1} \frac{\zeta_2^{p-1}}{p-1},$$

[ $\text{Exp}_p$  and  $\log_p$  denote the truncation of the exponential and the logarithm, respectively, by terms of degree  $> p-1$ ], and

$$Y \stackrel{\text{def}}{=} \frac{(\lambda X_1 + 1)^p - 1}{\lambda^p} = \frac{h(T)}{g(T)}.$$

Then  $f$  is a cyclic Galois cover of degree  $p^2$  which lifts the Galois cover  $f_k$  (cf. [Gr-Ma], the discussion in the beginning of 3, and Lemma 5.2).

We claim that  $Y$  is smooth over  $R$ . Indeed, the degree of the different in the morphism  $f_k : Y_k \rightarrow \tilde{X}_k$  in this case is

$$d_s = (m_1 + 1)(p-1)p + (p^2 m_1 - (p-1)m_1 + 1)(p-1).$$

Moreover, the above second equation (ii) defining the lifting  $f$  is

$$(X'_2)^p = (\lambda X_1 + 1) \text{Exp}_p(\mu^p Y) = (1 + \lambda X_1) \left( 1 + \mu^p \frac{h(T)}{g(T)} + \dots + \frac{\mu^{p(p-1)} h(T)^{(p-1)}}{(p-1)! g(T)^{(p-1)}} \right)$$

and

$$1 + \mu^p \frac{h(T)}{g(T)} + \dots + \frac{\mu^{p(p-1)} h(T)^{(p-1)}}{(p-1)! g(T)^{(p-1)}}$$

equals

$$\frac{(p-1)!g(T)^{p-1} + \mu^p(p-1)!h(T)g(T)^{p-2} + \dots + \mu^{p(p-1)}h(T)^{(p-1)}}{(p-1)!g(T)^{p-1}}.$$

Furthermore,

$$(p-1)!g(T)^{p-1} + \mu^p(p-1)!h(T)g(T)^{p-2} + \dots + \mu^{p(p-1)}h(T)^{(p-1)}$$

can be written as a series in  $X_1^{-\frac{1}{m_1}}$ , whose Weierstrass degree is  $pm_1(p-1)$  [since we assumed the degree of  $g(T)$  to be  $m_1$ ]. From this we deduce that the degree of the generic different  $d_\eta$  in the cover  $f_K : Y_K \rightarrow \tilde{X}_K$  satisfies

$$d_\eta \leq (m_1 + 1)(p^2 - 1) + pm_1(p-1)^2,$$

which implies  $d_\eta \leq d_s$ . One then concludes that  $d_\eta = d_s$ , hence that  $Y$  is smooth over  $R$ , since in general we must have  $d_s \leq d_\eta$ . Moreover, we have [by construction] a natural factorisation  $g : Y \rightarrow Y' \xrightarrow{h} \tilde{X}$ .  $\square$

**Lemma 3.2.2.** *With the same notations as above. Assume that  $g(T) = T^{m_1}$ . [Thus, in particular,  $h(T) \in R[[T]]$  is a unit]. [This case is rather special, since the corresponding smooth lifting  $h$  of the Galois sub-cover  $h_k$  has the property that all branched points are equidistant in the  $p$ -adic topology of  $K$ ]. Then there exists a smooth lifting  $f$  of  $f_k$  which dominates the smooth lifting  $h$  of  $h_k$ . In particular, [Conj-O4-rev] is true under these conditions for the Galois cover  $f_k$ , and the smooth lifting  $h$  of the sub-cover  $h_k$ .*

*Proof.* Consider the lifting

$$f : Y \rightarrow \tilde{X}$$

of the Galois cover  $f_k : Y_k \rightarrow \tilde{X}_k$ , which is generically given by the equations

$$(i') \quad \frac{(\lambda X_1 + 1)^p - 1}{\lambda^p} = f(T),$$

where

$$f(T) = \frac{h(T)}{T^{m_1}}$$

satisfies the above condition in this Lemma, and

$$(ii') \quad [\lambda X_2 + \text{Exp}_p(\mu X_1) \left( 1 + \sum_{\substack{0 \leq j < m_1 \\ 0 < i < p}} T^{-j} \mu^i (p-i)! P_{j,p-i}(g(T)) \right)]^p$$

$$= (G(T^{-1}) + p\mu^p \sum_{0 < s < r} A_s T^{-s})(\lambda X_1 + 1),$$

where

$$\text{Exp}_p X \stackrel{\text{def}}{=} 1 + X + \dots + \frac{X^{p-1}}{(p-1)!}$$

is the truncated exponential, and

$$\mu \stackrel{\text{def}}{=} \log_p(\zeta_2) = 1 - \zeta_2 + \dots + (-1)^{p-1} \frac{\zeta_2^{p-1}}{p-1},$$

are as in the proof of Lemma 3.2.1 above, the polynomial

$$G \stackrel{\text{def}}{=} G\left(\frac{(\lambda X_1 + 1)^p - 1}{\lambda^p}\right)$$

is defined in a similar way as in [Gr-Ma], Lemma 5.4,  $P_{j,p-i} \in R[X]$  are primitive polynomials which lift the  $p_{j,p-i} \in k[x]$ , and  $A_s \in R$  lift the  $a_s$  (cf. loc. cit). Then  $f : \tilde{Y} \rightarrow X$  is a Galois cover with a cyclic Galois group [isomorphic to  $\mathbb{Z}/p^2\mathbb{Z}$ ] and  $Y$  is smooth over  $R$ , as follows from the local criterion for good reduction (cf. [Gr-Ma], 3.4), by using Lemma 5.4 in [Gr-Ma] [where among others the degree of  $G$  in  $T^{-1}$  is computed], and the same argument as in the proof of Theorem 5.5 in loc. cit. [The key points here are that  $X_1^{-\frac{1}{m_1}}$  is a parameter for the disc  $Y'$ , and the key Lemma 5.4 in [Gr-Ma] is valid by replacing  $G \stackrel{\text{def}}{=} G(T^{-m_1})$  there by  $G \stackrel{\text{def}}{=} G(f(T))$  in our case (formally speaking only the degree in  $T^{-1}$  of  $f(T)$ , which is  $m_1$ , plays a role in loc. cit)]. Moreover, we have [by construction] a natural factorisation  $g : Y \rightarrow Y' \xrightarrow{h} \tilde{X}$ .  $\square$

**3.3.** Next, we will introduce the notion of fake liftings of cyclic Galois covers between curves. We will work within the framework of [Conj-O2-Rev].

Let  $n \geq 1$  be a positive integer. Let

$$f_k : Y_k \rightarrow \mathbb{P}_k^1$$

be a finite ramified Galois cover, where  $Y_k$  is a smooth  $k$ -curve, with Galois group  $G \xrightarrow{\sim} \mathbb{Z}/p^n\mathbb{Z}$  a cyclic  $p$ -group with cardinality  $p^n$ . We denote by

$$g_k : X_k \rightarrow \mathbb{P}_k^1$$

the unique sub-cover of  $f_k$  which is Galois with Galois group

$$H \xrightarrow{\sim} \mathbb{Z}/p^{n-1}\mathbb{Z}.$$

We have a canonical factorisation

$$f_k : Y_k \xrightarrow{h_k} X_k \xrightarrow{g_k} \mathbb{P}_k^1,$$

where  $h_k : Y_k \rightarrow X_k$  is a cyclic Galois cover between smooth  $k$ -curves of degree  $p$ .

We assume that the Galois cover  $g_k : X_k \rightarrow \mathbb{P}_k^1$  can be lifted to a Galois cover between smooth  $R$ -curves [in other words admits a smooth lifting over  $R$  (cf. Definition 2.5.2)], i.e. there exists a finite Galois cover

$$g : \mathcal{X} \rightarrow \mathbb{P}_R^1$$

with Galois group  $H$ , where  $\mathcal{X}$  is smooth over  $R$ ,  $\mathcal{X}_k \stackrel{\text{def}}{=} \mathcal{X} \times_R k$  is isomorphic to  $X_k$ , and such that the morphism induced by  $g$  at the level of special fibers

$$g_k : \mathcal{X}_k \rightarrow \mathbb{P}_k^1,$$

is isomorphic to the Galois cover  $g_k : X_k \rightarrow \mathbb{P}_k^1$ .

By Theorem 2.5.3 there exists a Garuti lifting (cf. Definition 2.5.2) of the Galois cover  $f_k$  which dominates  $g$ . We assume [for simplicity] that such a Garuti lifting is defined over  $R$ , i.e. there exists a finite Galois cover

$$\tilde{f} : \mathcal{Y} \rightarrow \mathbb{P}_R^1$$

with Galois group  $G$ , and  $\mathcal{Y}$  normal, which dominates  $g$ , i.e. we have a factorisation

$$\tilde{f} : \mathcal{Y} \xrightarrow{\tilde{h}} \mathcal{X} \xrightarrow{g} \mathbb{P}_R^1,$$

and such that the morphism

$$\tilde{f}_k : \mathcal{Y}_k \stackrel{\text{def}}{=} \mathcal{Y} \times_R k \rightarrow \mathbb{P}_k^1$$

between special fibers is generically étale, Galois with Galois group  $G$ , dominates  $g_k$  [i.e. we have a factorisation  $\tilde{f}_k : \mathcal{Y}_k \rightarrow X_k \xrightarrow{g_k} \mathbb{P}_k^1$ ], the normalisation  $\mathcal{Y}_k^{\text{nor}}$  of  $\mathcal{Y}_k$  is isomorphic to  $Y_k$  [in particular,  $\mathcal{Y}_k$  is irreducible], and the natural morphism between the normalisations

$$\mathcal{Y}_k^{\text{nor}} \rightarrow \mathbb{P}_k^1$$

[which is Galois] is isomorphic to  $f_k$ .

Let  $\delta_\eta \stackrel{\text{def}}{=} \delta_{\tilde{f}_K}$  (resp.  $\delta_s \stackrel{\text{def}}{=} \delta_{f_k}$ ) be the degree of the different in the morphism  $\tilde{f}_K : \mathcal{Y}_K \stackrel{\text{def}}{=} \mathcal{Y} \times_R K \rightarrow \mathbb{P}_K^1$  between generic fibres (resp. in the morphism  $f_k : Y_k \rightarrow \mathbb{P}_k^1$ ). It is well-known [and easy to verify] that we have the inequality

$$\delta_\eta \geq \delta_s.$$

Furthermore, the equality

$$\delta_\eta = \delta_s$$

holds if and only if  $\mathcal{Y}$  is smooth over  $R$  [which is equivalent to  $\mathcal{Y}_k$  being isomorphic to  $Y_k$ ], as follows from the local criterion for good reduction (cf. [Gr-Ma], 3.4).

We will consider the following assumption.

**3.3.1 Assumption (A).** Let  $n \geq 1$  be a positive integer, and  $f_k : Y_k \rightarrow \mathbb{P}_k^1$  a cyclic Galois cover with Galois group  $G \xrightarrow{\sim} \mathbb{Z}/p^n\mathbb{Z}$ , with  $Y_k$  a smooth  $k$ -curve. Let  $g_k : X_k \rightarrow \mathbb{P}_k^1$  be the unique Galois sub-cover of  $f_k$  of degree  $p^{n-1}$ . Assume that  $g_k$  has a smooth Galois lifting  $g : \mathcal{X} \rightarrow \mathbb{P}_{R'}^1$  [over some finite extension  $R'/R$ ] (cf. Definition 2.5.2).

We say that the Galois cover  $f_k : Y_k \rightarrow \mathbb{P}_k^1$  satisfies the assumption (A), with respect to the smooth lifting  $g$  of the sub-cover  $g_k$ , if for all possible Garuti liftings  $\tilde{f} : \mathcal{Y} \rightarrow \mathbb{P}_{R''}^1$  of the Galois cover  $f_k : Y_k \rightarrow \mathbb{P}_k^1$  which dominate  $g$  [see preceding discussion], and are defined over a finite extension  $R''/R'$  [the existence of such an  $\tilde{f}$  is guaranteed by Theorem 2.5.3], the strict inequality

$$\delta_\eta \stackrel{\text{def}}{=} \delta_{\tilde{f}_{K''}} > \delta_s \stackrel{\text{def}}{=} \delta_{f_k}$$

[where  $K'' \stackrel{\text{def}}{=} \text{Fr}(R'')$ ] holds.

In other words the assumption (A) is satisfied if there doesn't exist a smooth lifting of  $f_k$  which dominates the given smooth lifting  $g$  of the sub-cover  $g_k$  of  $f_k$ .

Note that if the above revisited version of Oort's conjecture [**Conj-O2-Rev**] (cf Remark 3.1.2) is true then no Galois cover  $f_k : Y_k \rightarrow \mathbb{P}_k^1$  as above satisfies the assumption (A).

Next, we introduce the notion of fake liftings of cyclic Galois covers between curves, which naturally arise if cyclic Galois covers satisfy the above assumption (A).

**Definition 3.3.2 (Fake liftings of Cyclic Covers between Curves).** Assume that the Galois cover  $f_k : Y_k \rightarrow \mathbb{P}_k^1$  satisfies the assumption (A), with respect to the smooth lifting  $g$  of the sub-cover  $g_k$  (cf. 3.3.1). Let

$$\delta \stackrel{\text{def}}{=} \min\{\delta_{\tilde{f}_{K''}}\},$$

where the minimum is taken among all possible Garuti liftings  $\tilde{f} : \mathcal{Y} \rightarrow \mathbb{P}_{R''}^1$  of  $f_k$  as above, which dominate the smooth lifting  $g : \mathcal{X} \rightarrow \mathbb{P}_{R'}^1$  of the sub-cover  $g_k : X_k \rightarrow \mathbb{P}_k^1$ . [Note that  $\delta > \delta_s$  by assumption].

We call a lifting  $\tilde{f} : \mathcal{Y} \rightarrow \mathbb{P}_{R''}^1$  as above satisfying the equality

$$\delta_{\tilde{f}_{K''}} = \delta$$

a fake lifting of the Galois cover  $f_k : Y_k \rightarrow \mathbb{P}_k^1$ , relative to the smooth lifting  $g$  of the sub-cover  $g_k$ . Note that if  $\tilde{f} : \mathcal{Y} \rightarrow \mathbb{P}_{R''}^1$  is a fake lifting of the Galois cover  $f_k$  then  $\mathcal{Y}$  is [by definition] not smooth over  $R''$ .

**Remark 3.3.3.** Fake liftings as in Definition 3.3.2 won't exist if the revisited Oort conjecture [**Conj-O2-Rev**] is true, hence the reason we call them fake. Moreover, in order to prove the [revisited] Oort conjecture it suffices to prove that fake liftings do not exist, as follows from the various definitions above.

**3.4.** In this sub-section we introduce some terminology related to the semi-stable geometry of curves, which will be used in the next sub-section 3.5, where we investigate the geometry of the [minimal] semi-stable models of fake liftings of cyclic Galois [of  $p$ -power order] covers between smooth curves.

Let  $f_k : Y_k \rightarrow \mathbb{P}_k^1$  be a finite ramified Galois cover with Galois group  $G \xrightarrow{\sim} \mathbb{Z}/p^n\mathbb{Z}$  a cyclic group of order  $p^n$ ,  $n \geq 1$ . Let  $G \twoheadrightarrow H \xrightarrow{\sim} \mathbb{Z}/p^{n-1}\mathbb{Z}$  be the [unique] quotient of  $G$  with cardinality  $p^{n-1}$ . Let  $g_k : X_k \rightarrow \mathbb{P}_k^1$  be the cyclic sub-cover of  $f_k$  with Galois group  $H$ . Assume that there exists  $g : \mathcal{X} \rightarrow \mathbb{P}_R^1$  a smooth Galois lifting of  $g_k$  over  $R$  (cf. Definition 2.5.2). Let  $\tilde{f} : \mathcal{Y} \rightarrow \mathbb{P}_R^1$  be a fake lifting of the Galois cover  $f_k : Y_k \rightarrow \mathbb{P}_k^1$  [with respect to the smooth lifting  $g$  of  $g_k$ ], which dominates the smooth lifting  $g$  of  $g_k$  (cf. Definition 3.3.2). [We assume that both  $\tilde{f}$  and  $g$  are defined over  $R$  for simplicity]. We have a natural factorization  $\tilde{f} : \mathcal{Y} \xrightarrow{h} \mathcal{X} \xrightarrow{g} \mathbb{P}_R^1$  where  $h : \mathcal{Y} \rightarrow \mathcal{X}$  is a finite Galois cover of degree  $p$ , with  $\mathcal{Y}$  normal and non smooth over  $R$ .

Next, we assume that  $\mathcal{Y}$  admits a semi-stable model over  $R$ . [It follows from the semi-stable reduction theorem for curves (cf. [De-Mu], and [Ab1]) that  $\mathcal{Y}$  admits a semi-stable model after eventually a finite extension of  $R$ ]. More precisely, we assume that there exists a birational morphism

$$\sigma : \mathcal{Y}' \rightarrow \mathcal{Y}$$

with  $\mathcal{Y}'$  semi-stable, i.e. the special fiber  $\mathcal{Y}'_k \stackrel{\text{def}}{=} \mathcal{Y}' \times_R k$  of  $\mathcal{Y}'$  is reduced, and its only singularities are ordinary double points. We also assume that the ramified points in the morphism  $\tilde{f}_K : \mathcal{Y}_K \rightarrow \mathbb{P}_K^1$  specialise in smooth distinct points of  $\mathcal{Y}'_k$ . Moreover, we will assume that the birational morphism  $\sigma$  is minimal with respect to the above properties. In particular, the action of the Galois group  $G \xrightarrow{\sim} \mathbb{Z}/p^n\mathbb{Z}$  on  $\mathcal{Y}$  extends to an action of  $G$  on  $\mathcal{Y}'$ . Let

$$\mathcal{P} \stackrel{\text{def}}{=} \mathcal{Y}'/G$$

be the quotient of  $\mathcal{Y}'$  by  $G$ , and

$$\tilde{f}' : \mathcal{Y}' \rightarrow \mathcal{P}$$

the natural morphism [which is Galois with Galois group  $G$ ]. Let

$$\tilde{g} : \mathcal{X}' \rightarrow \mathcal{P}$$

be the unique sub-cover of  $\tilde{f}'$  which is Galois with Galois group  $H$  [ $\mathcal{X}'$  is the quotient of  $\mathcal{Y}'$  by the unique subgroup of  $G$  with cardinality  $p$ ]. Then  $\mathcal{P}$  and  $\mathcal{X}'$  are semi-stable  $R$ -curves (cf. [Ra], appendice), and we have the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{Y} & \xrightarrow{h} & \mathcal{X} & \xrightarrow{g} & \mathbb{P}_R^1 \\ \sigma \uparrow & & \uparrow & & \uparrow \\ \mathcal{Y}' & \xrightarrow{\tilde{h}} & \mathcal{X}' & \xrightarrow{\tilde{g}} & \mathcal{P} \end{array}$$

where the vertical maps are birational morphisms, and the horizontal maps are finite morphisms.

To the special fiber  $\mathcal{Y}'_k \stackrel{\text{def}}{=} \mathcal{Y}' \times_R k$  of  $\mathcal{Y}'$  [which is a semi-stable  $k$ -curve] one associates a graph  $\Gamma$  whose vertices

$$\text{Ver}(\Gamma) \stackrel{\text{def}}{=} \{Y_i\}_{i=0}^{n'}$$

are the irreducible components of  $\mathcal{Y}'_k$ , and edges are the double points

$$\text{Edg}(\Gamma) \stackrel{\text{def}}{=} \{y_j\}_{j \in J}$$

of  $\mathcal{Y}'_k$ . A double point  $y_j \in Y_t \cap Y_s$  defines an edge linking the vertices  $Y_t$  and  $Y_s$ . We assume that  $Y_0$  is the strict transform of  $\mathcal{Y}_k$  [which is irreducible] in  $\mathcal{Y}'$ .

One also associates to the special fibre  $\mathcal{X}'_k \stackrel{\text{def}}{=} \mathcal{X}' \times_R k$  of  $\mathcal{X}'$  [which is a semi-stable  $k$ -curve] a graph  $\Gamma'$  whose vertices

$$\text{Ver}(\Gamma') \stackrel{\text{def}}{=} \{X_i\}_{i=0}^m$$

are the irreducible components of  $\mathcal{X}'_k$ , and edges are the double points

$$\text{Edg}(\Gamma') \stackrel{\text{def}}{=} \{x_j\}_{j \in J'}$$

of  $\mathcal{X}'_k$ . We assume that  $X_0$  is the strict transform of  $\mathcal{X}_k \xrightarrow{\sim} X_k$  in  $\mathcal{X}'$ . Then it follows easily [from the fact that  $\mathcal{X}$  is smooth] that the graph  $\Gamma'$  is a tree, and all the irreducible components of  $\mathcal{X}'_k$  which are distinct from  $X_0$  are isomorphic to  $\mathbb{P}^1_k$ . We choose an orientation of  $\Gamma'$  starting from  $X_0$  towards the end vertices of the tree  $\Gamma'$ . We have a natural morphism of graphs

$$\Gamma \rightarrow \Gamma'.$$

Similarly one associates to the special fibre  $\mathcal{P}_k \stackrel{\text{def}}{=} \mathcal{P} \times_R k$  of  $\mathcal{P}$  [which is a semi-stable  $k$ -curve] a graph  $\Gamma''$  whose vertices

$$\text{Ver}(\Gamma'') \stackrel{\text{def}}{=} \{P_i\}_{i=0}^n$$

are the irreducible components of  $\mathcal{P}_k$ , and edges are the double points

$$\text{Edg}(\Gamma'') \stackrel{\text{def}}{=} \{\tilde{x}_j\}_{j \in J''}$$

of  $\mathcal{P}_k$ . We assume that  $P_0$  is the strict transform of  $\mathbb{P}^1_k$  [the special fibre of  $\mathbb{P}^1_R$ ] in  $\mathcal{P}$ . The graph  $\Gamma''$  is a tree and all the irreducible components of  $\mathcal{P}_k$  are isomorphic to  $\mathbb{P}^1_k$ . We choose an orientation of  $\Gamma''$  starting from  $P_0$  towards the end vertices of the tree  $\Gamma''$ . We have natural morphisms of graphs

$$\Gamma \rightarrow \Gamma' \rightarrow \Gamma''.$$

The morphism  $\Gamma \rightarrow \Gamma''$  (resp.  $\Gamma' \rightarrow \Gamma''$ ) is  $G$ -equivariant (resp.  $H$ -equivariant). [The graph  $\Gamma$  (resp.  $\Gamma'$ ) is naturally endowed with an action of the group  $G$  (resp.  $H$ )].

Let  $Y_i$  be a vertex of the graph  $\Gamma$ . To  $Y_i$  one associates two subgroups of the Galois group  $G \xrightarrow{\sim} \mathbb{Z}/p^n\mathbb{Z}$  of the Galois cover  $\tilde{f} : \mathcal{Y} \rightarrow \mathbb{P}^1_R$ : the decomposition subgroup  $D_i \subseteq G$ , and the inertia subgroup  $I_i \subseteq D_i$ , at the generic point of  $Y_i$  in the Galois cover  $\tilde{f}$ . We call the [irreducible component] vertex  $Y_i$  of  $\Gamma$  an end vertex [or end component] of  $\Gamma$  if the graph  $\Gamma$  is a tree, and if  $Y_i$  is an end vertex of this tree. We call  $Y_i$  a separable vertex of  $\Gamma$  if the inertia subgroup  $I_i$  which is associated to  $Y_i$  is trivial. Finally, we call the irreducible component  $Y_i$  a ramified vertex if

there exists a ramified point in the morphism  $f_K : \mathcal{Y}_K \rightarrow \mathbb{P}_K^1$  which specialises in the component  $Y_i$ .

Similarly let  $X_i$  be a vertex of the graph  $\Gamma'$ . To  $X_i$  one associates two subgroups of the Galois group  $H \xrightarrow{\sim} \mathbb{Z}/p^{n-1}\mathbb{Z}$  of the Galois cover  $g : \mathcal{X} \rightarrow \mathbb{P}_R^1$ : the decomposition subgroup  $\tilde{D}_i \subseteq H$ , and the inertia subgroup  $\tilde{I}_i \subseteq \tilde{D}_i$ , at the generic point of  $X_i$  in the Galois cover  $g$ . We call the vertex  $X_i$  of  $\Gamma'$  an end vertex of  $\Gamma'$  if  $X_i$  is an end vertex of the tree  $\Gamma'$ . We call  $X_i$  an internal vertex of  $\Gamma'$  if  $X_i$  is distinct from  $X_0$ , and the end vertices of  $\Gamma'$ . We call  $X_i$  a separable vertex of  $\Gamma'$  if the inertia subgroup  $\tilde{I}_i$  which is associated to  $X_i$  is trivial. Finally, we call the irreducible component  $X_i$  a ramified vertex if there exists a ramified point in the morphism  $g_K : \mathcal{X}_K \rightarrow \mathbb{P}_K^1$  which specialises in the component  $X_i$ .

Finally, By a geodesic in a finite tree linking two vertices we mean the path, or sub-tree, with smallest length which links the two vertices.

**3.5.** In this sub-section we first establish in the next Proposition some properties of the [not necessarily minimal] semi-stable model  $\mathcal{X}' \rightarrow \mathcal{X}$  of the smooth lifting  $g : \mathcal{X} \rightarrow \mathbb{P}_R^1$  of the Galois sub-cover  $g_k : X_k \rightarrow \mathbb{P}_k^1$  of  $f_k : Y_k \rightarrow \mathbb{P}_k^1$ .

**Proposition 3.5.1.** *let  $g_k : X_k \rightarrow \mathbb{P}_k^1$  be a finite ramified Galois cover with Galois group  $H \xrightarrow{\sim} \mathbb{Z}/p^{n-1}\mathbb{Z}$  [ $n > 1$ ], and  $X_k$  a smooth  $k$ -curve. Let  $g : \mathcal{X} \rightarrow \mathbb{P}_R^1$  be a smooth Galois lifting of  $g_k$  over  $R$  (cf. Definition 2.5.2). Assume that there exists a birational morphism  $\mathcal{X}' \rightarrow \mathcal{X}$  such that  $\mathcal{X}'$  is semi-stable, the action of  $H$  on  $\mathcal{X}$  extends to an action on  $\mathcal{X}'$ , and the ramified points in the Galois cover  $g_K : \mathcal{X}_K \rightarrow \mathbb{P}_K^1$  specialise in smooth distinct points of  $\mathcal{X}'_k$ . [We do not assume that  $\mathcal{X}'$  is minimal with respect to the above properties]. Let  $\mathcal{P} \stackrel{\text{def}}{=} \mathcal{X}'/H$  be the quotient of  $\mathcal{X}'$  by  $H$ . We have a commutative diagram:*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{g} & \mathbb{P}_R^1 \\ \uparrow & & \uparrow \\ \mathcal{X}' & \xrightarrow{\tilde{g}} & \mathcal{P} \end{array}$$

where  $\mathcal{P}$  is a semi-stable  $R$ -curve, and the vertical maps are birational morphisms.

Let  $\Gamma'$  (resp.  $\Gamma''$ ) be the graph associated to the semi-stable  $k$ -curve  $\mathcal{X}'_k$  (resp.  $\mathcal{P}_k$ ). Let  $\text{Ver}(\Gamma') \stackrel{\text{def}}{=} \{X_i\}_{i=0}^m$  (resp.  $\text{Ver}(\Gamma'') \stackrel{\text{def}}{=} \{P_i\}_{i=0}^{n'}$ ) be the set of vertices of  $\Gamma'$  (resp. of  $\Gamma''$ ). Then we have a natural morphism  $\Gamma' \rightarrow \Gamma''$  of graphs and the followings hold.

(i) The graphs  $\Gamma'$  and  $\Gamma''$  are trees. Furthermore, each vertex  $X_i$  (resp.  $P_i$ ) of  $\Gamma'$  (resp. of  $\Gamma''$ ) which is distinct from the strict transform of  $\mathcal{X}_k$  (resp. distinct from the strict transform of the special fibre of  $\mathbb{P}_R^1$ ) is isomorphic to  $\mathbb{P}_k^1$ .

Let  $X_0$  be the strict transform of  $\mathcal{X}_k \xrightarrow{\sim} X_k$  in  $\mathcal{X}'$ . We choose an orientation of the tree  $\Gamma'$  starting from  $X_0$  towards the end vertices of  $\Gamma'$ . For a vertex  $X_i$  of  $\Gamma'$  we will denote by  $\tilde{D}_i$  (resp.  $\tilde{I}_i \subseteq \tilde{D}_i$ ) the decomposition (resp. inertia) subgroup of  $H$  at the generic point of  $X_i$ . Then:

(ii)  $\tilde{D}_0 = H$  and  $\tilde{I}_0 = \{1\}$ .

(iii) Let  $X_i$  be an internal vertex of  $\Gamma'$  [i.e.  $X_i$  is distinct from  $X_0$  and from the end vertices of  $\Gamma'$ ], and  $X_j$  an adjacent vertex to  $X_i$  in the direction moving towards the end vertices of  $\Gamma'$ . Then the following two cases occur:

(1) Either  $\tilde{D}_i = \tilde{I}_i$ . In this case  $\tilde{D}_j = \tilde{D}_i$ .



(2) Or  $\tilde{I}_i \subsetneq \tilde{D}_i$ . In this case  $\tilde{D}_j = \tilde{I}_i$  and we have an exact sequence

$$1 \rightarrow \tilde{D}_j \rightarrow \tilde{D}_i \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow 0.$$

Furthermore, in the case (2) if  $\tilde{X}_i$  denotes the image of  $X_i$  in the quotient  $\mathcal{X}'/\tilde{I}_i$  of  $\mathcal{X}'$  by  $\tilde{I}_i$  then the natural morphism  $\tilde{X}_i \rightarrow P_i$ , where  $P_i \xrightarrow{\sim} \mathbb{P}_k^1$  is the image of  $X_i$  in  $\Gamma''$ , is a Galois cover of degree  $p$  ramified above a unique point  $\infty \in P_i$  [which is the edge of the geodesic linking  $P_i$  to  $P_0$ , which is linked to  $P_i$ ] with Hasse conductor  $m = 1$  at  $\infty$ .

In particular, when we move in the graph  $\Gamma'$  starting from  $X_0$  towards the end vertices of  $\Gamma'$  then the cardinality of the decomposition group  $\tilde{D}_i$  (resp. the cardinality of the inertia subgroup  $\tilde{I}_i$ ) of a vertex  $X_i$  decreases. More precisely, if when moving from a vertex  $X_i$  towards the end vertices of  $\Gamma'$  we encounter a vertex  $X_j$  then  $\tilde{D}_j \subseteq \tilde{D}_i$  and  $\tilde{I}_j \subseteq \tilde{I}_i$ .

(iv) Let  $X_i$  be a separable vertex of  $\Gamma'$  [i.e.  $\tilde{I}_i = \{1\}$ ] which is distinct from  $X_0$ . Then either  $X_i$  is an internal vertex [of  $\Gamma'$ ] which is adjacent to an end vertex of the graph  $\Gamma'$ . Furthermore,  $\tilde{D}_i = \mathbb{Z}/p\mathbb{Z}$  in this case and  $X_i$  is a Galois cover of  $\mathbb{P}_k^1$  ramified above a unique point  $\infty \in \mathbb{P}_k^1$  with Hasse conductor  $m = 1$  at  $\infty$ . [In this case if  $X_j$  is the end vertex of  $\Gamma'$  which is adjacent to  $X_i$  then  $\tilde{D}_j = \{1\}$  (cf. (ii), (2))]. Or,  $X_i$  is an end vertex of  $\Gamma'$ , and two cases can occur: either  $\tilde{D}_i = \mathbb{Z}/p\mathbb{Z}$  and  $X_i$  is a Galois cover of  $\mathbb{P}_k^1$  ramified above a unique point  $\infty \in \mathbb{P}_k^1$  [which is the point linking  $X_i$  to the rest of the tree  $\Gamma'$ ] with Hasse conductor  $m = 1$  at  $\infty$ , or  $\tilde{D}_i = \{1\}$  and  $X_i$  is adjacent to a [unique] internal separable vertex  $X_j$  with  $\tilde{D}_j \xrightarrow{\sim} \mathbb{Z}/p\mathbb{Z}$ ,  $\tilde{I}_j = \{1\}$ , and  $X_j$  is a Galois cover of  $\mathbb{P}_k^1$  ramified above a unique point  $\infty \in \mathbb{P}_k^1$  [which is the edge of the geodesic linking  $P_j$  to  $P_0$ , which is linked to  $P_j$ ] with Hasse conductor  $m = 1$  at  $\infty$ .

Let  $0 < j \leq n-1$  be an integer. Let  $x \in \mathcal{X}_K$  be a ramified point in the morphism  $g_K : \mathcal{X}_K \rightarrow \mathbb{P}_K^1$ . We say that the ramified point  $x$  is of type  $j$  if the inertia subgroup  $\tilde{I}_x \subseteq H$  at  $x$  is isomorphic to  $\mathbb{Z}/p^j\mathbb{Z}$ . A vertex  $X_i$  of  $\Gamma'$  is called a ramified vertex of type  $j$  if there exists a ramified point  $x$  of type  $j$  in the morphism  $g_K : \mathcal{X}_K \rightarrow \mathbb{P}_K^1$  which specialises in the component  $X_i$ .

(v) Let  $X_i$  be a ramified component of  $\Gamma'$ . Then  $X_i$  is of type  $j$  for a unique integer  $0 < j \leq n-1$ . In other words if  $0 < j < j' \leq n-1$  are integers then ramified points  $x \in \mathcal{X}_K$  (resp.  $x' \in \mathcal{X}_K$ ) of type  $j$  (resp. type  $j'$ ) in the morphism  $g_K : \mathcal{X}_K \rightarrow \mathbb{P}_K^1$  specialise in distinct irreducible components of  $\mathcal{X}_k$ . More precisely, if  $X_i$  is a ramified vertex of type  $j$  then the inertia subgroup  $\tilde{I}_i$  which is associated to  $X_i$  has cardinality  $p^j$ , i.e.  $\tilde{I}_i \xrightarrow{\sim} \mathbb{Z}/p^j\mathbb{Z}$ . [In other words the type  $j$  of a ramified component  $X_i$  is uniquely determined by  $X_i$ ].

Furthermore, let  $P_i$  be the image of  $X_i$  in  $\mathcal{P}$ . Then the natural morphism  $X_i \rightarrow P_i$  has the structure of a  $\mu_{p^j}$ -torsor outside the double points supported by  $P_i$ , and the specialisation of the branched points in  $P_i$  [in this case  $\tilde{D}_i = \tilde{I}_i$ ].

(vi) Let  $X_i$  be a ramified vertex of  $\mathcal{X}_k$  of type  $j$ . Then when moving in the graph  $\Gamma'$  from  $X_i$  towards the end vertices of  $\Gamma'$  we encounter at most a unique ramified vertex  $X_{i'} \neq X_i$ . Moreover, in such a component  $X_{i'}$  specialises a unique ramified point in the morphism  $f_K : \mathcal{X}_K \rightarrow \mathbb{P}_K^1$ , and the component  $X_{i'}$  is necessarily of the same type  $j$  as  $X_i$ . [In other words the graph  $\Gamma'$  separates the directions of the ramified vertices of  $\Gamma'$  which are of distinct types].

(vii) Assume that  $\mathcal{X}$  is minimal [with respect to its defining properties above]. Then the ramified vertices in the graph  $\Gamma'$  are the end vertices of the tree  $\Gamma'$ .

*Proof.* Assertion (i) is clear and follows immediately from the fact that  $\mathcal{X}$  is smooth.

Assertion (ii) is also clear since  $\mathcal{X}_k$  is irreducible and the natural morphism  $\mathcal{X}_k \rightarrow \mathbb{P}_k^1$  [which is isomorphic to  $g_k : X_k \rightarrow \mathbb{P}_k^1$ ] is generically Galois with Galois group  $H$ .

Next, we prove (iii). Let  $X_i$  be an internal vertex of  $\Gamma'$ , and  $X_j$  an adjacent vertex to  $X_i$  in the direction moving towards the end vertices of  $\Gamma'$ . Let  $P_i$  (resp.  $P_j$ ) be the image of  $X_i$  (resp.  $X_j$ ) in  $\mathcal{P}$ .

Assume first that  $\tilde{D}_i = \tilde{I}_i$ , we will show that  $\tilde{D}_j = \tilde{D}_i$  in this case. Let  $\mathcal{X}_1 \stackrel{\text{def}}{=} \mathcal{X}'/\tilde{D}_i$  be the quotient of  $\mathcal{X}'$  by  $\tilde{D}_i$ . Then  $\mathcal{X}_1$  is a semi-stable  $R$ -curve, and the configuration of the special fibre  $(\mathcal{X}_1)_k$  of  $\mathcal{X}_1$  is a tree-like (cf. (i)). The natural morphism  $\mathcal{X}_1 \rightarrow \mathcal{P}$  is by assumption completely split above the irreducible component  $P_i$  of  $\mathcal{P}_k$ , hence [a fortiori] is also completely split above  $P_j$ . This shows that  $\tilde{D}_j \subseteq \tilde{D}_i$ . Assume that  $\tilde{D}_j \subsetneq \tilde{D}_i$ . Let  $x \stackrel{\text{def}}{=} X_i \cap X_j$  which is a double point of  $\mathcal{X}'$  and  $x' \stackrel{\text{def}}{=} P_i \cap P_j$  its image in  $\mathcal{P}$ . Let  $\mathcal{X}'' \stackrel{\text{def}}{=} \mathcal{X}'/\tilde{D}_j$  be the quotient of  $\mathcal{X}'$  by  $\tilde{D}_j$  [ $\mathcal{X}''$  is a semi-stable  $R$ -curve and the configuration of the special fibre  $\mathcal{X}''_k$  of  $\mathcal{X}''$  is a tree-like], and  $X''_i$  the image of  $X_i$  in  $\mathcal{X}''$ . The natural morphism  $\mathcal{X}'' \rightarrow \mathcal{P}$  is by assumption completely split above  $P_j$ , hence also completely split above the double point  $x'$ . In particular, the natural morphism  $X''_i \rightarrow P_i$  is étale above  $x'$  and is generically Galois with Galois group  $\tilde{D}_i/\tilde{D}_j$ . This contradicts the fact that  $\tilde{D}_i = \tilde{I}_i$ . Hence  $\tilde{D}_j = \tilde{D}_i$  necessarily.

Assume now that  $\tilde{I}_i \subsetneq \tilde{D}_i$  and write  $D'_i \stackrel{\text{def}}{=} \tilde{D}_i/\tilde{I}_i \neq \{1\}$ . Let  $\tilde{X}_i$  be the image of  $X_i$  in the quotient  $\mathcal{X}'/\tilde{I}_i$  of  $\mathcal{X}'$  by  $\tilde{I}_i$ . We have a natural morphism  $\tilde{X}_i \rightarrow P_i$  which is generically Galois with Galois group  $D'_i$ . The vertex  $P_i \in \text{Ver } \Gamma''$  is an internal vertex of the tree  $\Gamma''$  [as is easily seen since  $X_i$  is an internal vertex of  $\Gamma'$ ], hence is linked to more than one double point of  $\Gamma''$ . More precisely,  $P_i$  is linked to a unique double point  $x'$  which links  $P_i$  to the geodesic joining  $P_i$  and the vertex  $P_0$  [ $P_0$  is the image of  $X_0$  in  $\mathcal{P}$ ], and [at least another] other double points linking  $P_i$  to the geodesics joining  $P_i$  and some of the end vertices of the graph  $\Gamma''$ .

If the natural morphism  $\tilde{X}_i \rightarrow P_i$  is unramified above the double point  $x'$  then it is easy to see that this would introduce loops in the configuration of  $\Gamma'$  hence the later won't be a tree. Thus, the morphism  $\tilde{X}_i \rightarrow P_i$  must [totally] ramify above the double point  $x'$ . In particular, this morphism is necessarily unramified above the remaining double points linking  $P_i$  to the end vertices of  $\Gamma''$ . Indeed, for otherwise the genus of  $\tilde{X}_i$  [hence that of  $X_i$ ] would be  $> 0$ , since the degree of this morphism is a power of  $p$ , as follows easily from the Riemann-Hurwitz genus formula, and this would contradict the second assertion in (i)].

Also the degree of the morphism  $\tilde{X}_i \rightarrow P_i$  is necessarily equal to  $p$ , and this morphism is only ramified above the double point  $x'$  with Hasse conductor  $m = 1$  at  $x'$  [for otherwise the genus of  $\tilde{X}_i$  [hence that of  $X_i$ ] would be  $> 0$  for similar reasons as above]. This also shows that  $\tilde{D}_j \subset \tilde{I}_i$  [indeed, the natural morphism  $\mathcal{X}'/\tilde{I}_i \rightarrow \mathcal{P}$  is easily seen to be completely split above the component  $P_j$  which is the image of  $X_j$  in  $\mathcal{P}$ ], and that we have a natural exact sequence

$$1 \rightarrow \tilde{I}_i \rightarrow \tilde{D}_i \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow 0.$$

Now we show that  $\tilde{D}_j = \tilde{I}_i$ . Assume that  $\tilde{D}_j \subsetneq \tilde{I}_i$ . Let  $\tilde{\mathcal{X}}' \stackrel{\text{def}}{=} \mathcal{X}'/\tilde{D}_j$  (resp.  $\tilde{\mathcal{X}}'' \stackrel{\text{def}}{=} \mathcal{X}'/\tilde{I}_i$ ) be the quotient of  $\mathcal{X}'$  by  $\tilde{D}_j$  (resp. the quotient of  $\mathcal{X}'$  by  $\tilde{I}_i$ ), and  $\tilde{X}'_i$

(resp.  $\tilde{X}_i''$ ) the image of  $X_i$  in  $\tilde{\mathcal{X}}'$  (resp.  $\tilde{\mathcal{X}}''$ ). By assumption the natural morphism  $\tilde{X}_i' \rightarrow \tilde{X}_i''$  [which is of degree  $\geq p$ ] must be on the one hand a homeomorphism, and on the other hand completely split above the image of the double point  $x \stackrel{\text{def}}{=} X_i \cap X_j$ . This is a contradiction. Hence we necessarily have the equality  $\tilde{I}_i = \tilde{D}_j$ . This proves the assertions 1 and 2 in (iii). The remaining assertion in (iii) follows easily from this.

The assertion (iv) follows easily from (iii), and the fact that if in a generically Galois cover  $f : C \rightarrow \mathbb{P}_k^1$  with Galois group a cyclic  $p$ -group we have  $C \xrightarrow{\sim} \mathbb{P}_k^1$ , then  $f$  has necessarily degree  $p$  and is ramified above a unique point  $\infty \in \mathbb{P}_k^1$  with Hasse conductor  $m = 1$  [as follows easily from the Riemann-Hurwitz genus formula, and Artin-Schreier-Witt theory].

Next, we prove (v). Let  $0 < j \leq n - 1$  be an integer. Let  $x \in \mathcal{X}_K$  be a ramified point in the morphism  $g_K : \mathcal{X}_K \rightarrow \mathbb{P}_K^1$  of type  $j$  which specialises in the irreducible component  $X_i$  of  $\mathcal{X}_k'$ . We will show that  $\tilde{I}_i = \tilde{I}_x$ , where  $\tilde{I}_x \xrightarrow{\sim} \mathbb{Z}/p^j\mathbb{Z}$  is the inertia subgroup at  $x$ .

Let  $\mathcal{X}_2 \stackrel{\text{def}}{=} \mathcal{X}'/\tilde{I}_x$  be the quotient of  $\mathcal{X}'$  by  $\tilde{I}_x$ , and  $\tilde{X}_i$  the image of  $X_i$  in  $\mathcal{X}_2$ . The natural morphism  $X_i \rightarrow \tilde{X}_i$  is a radicial morphism, as follows from [Sa], Corollary 4.1.2, hence  $\tilde{I}_x \subset \tilde{I}_i$ . Assume that  $\tilde{I}_x \subsetneq \tilde{I}_i$ . Let  $\mathcal{X}_2' \stackrel{\text{def}}{=} \mathcal{X}'/\tilde{I}_i$ , and  $\tilde{X}_i'$  the image of  $X_i$  in  $\mathcal{X}_2'$ . The natural morphism  $\tilde{X}_i \rightarrow \tilde{X}_i'$  [which has degree bigger than 1] is by assumption on the one hand radicial, and on the other hand unramified above the image of the specialisation of the ramified point  $x$  in  $\tilde{X}_i'$ , which is a contradiction. Hence we necessarily have  $\tilde{I}_x = \tilde{I}_i$ . The last assertion in (v) follows from Lemma 3.5.5 (see end of §3), and the corresponding assertion in the case where  $G \xrightarrow{\sim} \mathbb{Z}/p\mathbb{Z}$  in [Sa], Corollary 4.1.2.

Assertion (vi) follows directly from the next Lemma 3.5.2, by passing to the quotient of  $\mathcal{X}'$  by the unique subgroup  $H'$  of  $H$  with cardinality  $p$ .

Next, we prove (vii). Assume that  $\mathcal{X}$  is minimal with respect to its defining properties. Let  $X_i$  be a ramified vertex of the tree  $\Gamma'$ . We will show that  $X_i$  is necessarily an end vertex of  $\Gamma'$ . Assume that  $X_i$  [which is distinct from  $X_0$ ] is an internal vertex of  $\Gamma'$ . Let  $X_{\tilde{i}}$  be an end vertex of  $\Gamma'$  which we encounter when moving in  $\Gamma'$  from  $X_i$  towards the end vertices of  $\Gamma'$ , and  $\gamma$  the geodesic linking  $X_i$  and  $X_{\tilde{i}}$ . All vertices of  $\gamma$  are projective lines (cf. (i)).

In  $\gamma$  there exists at most a unique vertex  $X_j \neq X_i$  which is a ramified vertex (cf. (vi)). All vertices of  $\gamma$  which are not ramified vertices can be contracted in  $\mathcal{X}$  without destroying the defining properties of  $\Gamma'$ . Thus, we deduce that  $\gamma$  contains a unique vertex which is distinct from  $X_i$ , namely  $X_j$ , and the later  $X_j = X_{\tilde{i}}$  is an end vertex of  $\Gamma$ . By (vi) the vertex  $X_j$  is of the same type as the vertex  $X_i$ , and there exists a unique ramified point in the morphism  $\mathcal{X}_K \rightarrow \mathbb{P}_K^1$  which specialises in [a smooth point of]  $X_j$ . The vertex  $X_j$  can also be contracted in a [smooth] point of  $\mathcal{X}'$  which is supported by  $X_i$  and in this point will specialise [after contracting  $X_j$ ] a unique ramified point, which doesn't destroy the defining properties of  $\mathcal{X}'$ . But this would contradict the minimality of  $\mathcal{X}'$ . Thus,  $X_i$  is necessarily a terminal vertex to start with.  $\square$

The following lemma is used in the proof of assertion (vi) in Proposition 3.5.1.

**Lemma 3.5.2.** *Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a finite Galois cover between smooth  $R$ -curves with Galois group  $H' \xrightarrow{\sim} \mathbb{Z}/p\mathbb{Z}$ , such that the morphism  $f_K : \mathcal{X}_K \rightarrow \mathcal{Y}_K$  between*

generic fibres is ramified. Assume that there exists a birational morphism  $\mathcal{X}' \rightarrow \mathcal{X}$  such that  $\mathcal{X}'$  is a semi-stable  $R$ -curve, the action of the Galois group  $H'$  on  $\mathcal{X}$  extends to an action of  $H'$  on  $\mathcal{X}'$ , and the ramified points in the morphism  $f_K : \mathcal{X}_K \rightarrow \mathcal{Y}_K$  specialise in smooth distinct points of  $\mathcal{X}'_k$ . Then the graph  $\Gamma'$  associated to the special fibre  $\mathcal{X}'_k$  of  $\mathcal{X}'$  is a tree. Let  $X_0$  be the strict transform of  $\mathcal{X}_k$  in  $\mathcal{X}'$ . Choose an orientation of  $\Gamma'$  starting from  $X_0$  towards the end vertices of  $\Gamma'$ .

Let  $X_i$  be a vertex of  $\Gamma'$ . Assume that  $X_i$  is a ramified vertex of  $\Gamma'$  [i.e. there exists a ramified point in the morphism  $f_K : \mathcal{X}_K \rightarrow \mathcal{Y}_K$  which specialises in  $X_i$ ]. Then when moving in the graph  $\Gamma'$  from  $X_i$  towards the end vertices we encounter at most a unique ramified vertex  $X_j \neq X_i$ . Moreover, in such a component  $X_j$  specialises a unique ramified point in the morphism  $f_K : \mathcal{X}_K \rightarrow \mathcal{Y}_K$ .

*Proof.* The fact that the graph  $\Gamma'$  is a tree follows immediately from the fact that  $\mathcal{X}$  is smooth over  $R$ . Let  $X_0$  be the strict transform of  $\mathcal{X}_k$  in  $\Gamma'$ . Let  $X_i$  be a ramified component of  $\mathcal{X}_k$ . Then  $X_i \neq X_0$  as follows from [Sa], Corollary 4.1.2. Thus,  $X_i$  is either an internal or an end component of  $\Gamma'$ . Assume that  $X_i$  is an internal component. Let  $X_j$  be an irreducible component of  $\Gamma'$  which is a ramified vertex and that we encounter when moving from  $X_i$  towards the end vertices of  $\Gamma'$ . We will show that only a unique ramified point in the morphism  $f_K : \mathcal{X}_K \rightarrow \mathcal{Y}_K$  specialises in such a component  $X_j$ , and that such a component is unique.

After eventually contracting all the irreducible components which form the vertices of the geodesics of  $\Gamma'$  which link  $X_i$  to the end vertices of  $\Gamma'$  we can assume that  $X_i$  is an end vertex of  $\Gamma'$ . The component  $X_j$  then contracts to a smooth point  $x$  of  $X_i$  [which is the specialisation of some ramified points in the morphism  $f_K : \mathcal{X}_K \rightarrow \mathcal{Y}_K$ ]. Let  $P_i$  be the image of  $X_i$  in the quotient  $\mathcal{Y}' \stackrel{\text{def}}{=} \mathcal{X}'/H'$  of  $\mathcal{X}'$  by  $H'$ , and  $y$  the image of  $x$  in  $\mathcal{Y}'$  which is a smooth point. The natural morphism  $X_i \rightarrow P_i$  is a  $\mu_p$ -torsor (cf. loc. cit). Furthermore, the natural morphism  $\hat{\mathcal{O}}_{\mathcal{X},x} \rightarrow \hat{\mathcal{O}}_{\mathcal{Y},y}$  between the formal completions at the smooth points  $x$  and  $y$  has a degeneration on the boundary of the formal completion  $\hat{\mathcal{O}}_{\mathcal{Y},y}$  of type  $(\mu_p, 0, h)$  (cf. [Sa], Corollary 4.1.2), and there is a unique ramified point which specialises in  $x$  (cf. loc. cit).  $\square$

Proposition 3.5.1 has the following local analog, which describes the geometry of a [minimal] semi-stable model of an order  $p^n$  automorphism of a  $p$ -adic open disc [over  $K$ ] without inertia at  $\pi$  (cf. [Gr-Ma], 1), and which was proven in [Gr-Ma1] in the case of an order  $p$ -automorphism. [Though we state our result in terms of Galois covers between formal germs of smooth curves].

**Proposition 3.5.3.** *let  $f : \tilde{\mathcal{X}} \stackrel{\text{def}}{=} \text{Spf } A \rightarrow \tilde{\mathcal{Y}} \stackrel{\text{def}}{=} \text{Spf } B$  be a Galois cover between connected formal germs of smooth  $R$ -curves (i.e.  $A \xrightarrow{\sim} B \xrightarrow{\sim} R[[T]]$ ) which is Galois with Galois group  $G \xrightarrow{\sim} \mathbb{Z}/p^n\mathbb{Z}$ ,  $n \geq 1$ , and such that the natural morphism  $f_k : \tilde{\mathcal{X}}_k \stackrel{\text{def}}{=} \text{Spec } A/\pi A \rightarrow \tilde{\mathcal{Y}}_k \stackrel{\text{def}}{=} \text{Spec } B/\pi B$  between special fibres is generically separable. Assume that there exists a birational morphism  $\tilde{\mathcal{X}}' \rightarrow \tilde{\mathcal{X}}$  such that the ramified points in the morphism  $f_K : \tilde{\mathcal{X}}_K \stackrel{\text{def}}{=} \text{Spec}(A \otimes_R K) \rightarrow \tilde{\mathcal{Y}}_K \stackrel{\text{def}}{=} \text{Spec}(B \otimes_R K)$  specialise in smooth distinct points of  $\tilde{\mathcal{X}}'_k$ , and the action of  $G$  on  $\tilde{\mathcal{X}}$  extends to an action of  $G$  on  $\tilde{\mathcal{X}}'$ . [We do not assume that  $\tilde{\mathcal{X}}'$  is minimal with respect to the above properties]. Let  $\tilde{\mathcal{Y}}' \stackrel{\text{def}}{=} \tilde{\mathcal{X}}'/G$  be the quotient of  $\tilde{\mathcal{X}}'$  by  $G$ . Then  $\tilde{\mathcal{Y}}'$  is semi-stable (cf.*

[Ra], Appendice). We have a commutative digram:

$$\begin{array}{ccc} \tilde{\mathcal{X}} & \xrightarrow{f} & \tilde{\mathcal{Y}} \\ \uparrow & & \uparrow \\ \tilde{\mathcal{X}}' & \xrightarrow{\tilde{f}} & \tilde{\mathcal{Y}}' \end{array}$$

where the vertical maps are birational morphisms.

Let  $\Gamma'$  (resp.  $\Gamma''$ ) be the graph associated to the special fibre  $\tilde{\mathcal{X}}'$  (resp.  $\tilde{\mathcal{Y}}'$ ). Let  $\text{Ver}(\Gamma') \stackrel{\text{def}}{=} \{X_i\}_{i=0}^m$  (resp.  $\text{Ver}(\Gamma'') \stackrel{\text{def}}{=} \{Y_i\}_{i=0}^{n'}$ ) be the set of vertices of  $\Gamma'$  (resp. of  $\Gamma''$ ). Then we have a natural morphism  $\Gamma' \rightarrow \Gamma''$  of graphs and the followings hold.

(i) The graphs  $\Gamma'$  and  $\Gamma''$  are trees. Furthermore, each vertex  $X_i$  (resp.  $Y_i$ ) of  $\Gamma'$  (resp. of  $\Gamma''$ ) which is distinct from the strict transform of [the generic point of]  $\tilde{\mathcal{X}}_k$  in  $\tilde{\mathcal{X}}'$  (resp. distinct from the strict transform of [the generic point of]  $\tilde{\mathcal{Y}}_k$  in  $\tilde{\mathcal{Y}}'$ ) is isomorphic to  $\mathbb{P}_k^1$ .

Let  $X_0$  be the strict transform of [the generic point of]  $\tilde{\mathcal{X}}_k$  in  $\tilde{\mathcal{X}}'$ . We choose an orientation of the tree  $\Gamma'$  starting from  $X_0$  towards the end vertices of  $\Gamma'$ . For a vertex  $X_i$  of  $\Gamma'$  we will denote by  $\tilde{D}_i$  (resp.  $\tilde{I}_i \subseteq \tilde{D}_i$ ) the decomposition (resp. inertia) subgroup of  $H$  at the generic point of  $X_i$ . Then:

(ii)  $\tilde{D}_0 = H$  and  $\tilde{I}_0 = \{1\}$ .

(iii) Let  $X_i$  be an internal vertex of  $\Gamma'$  [i.e.  $X_i$  is distinct from  $X_0$  and from the end vertices of  $\Gamma'$ ], and  $X_j$  an adjacent vertex to  $X_i$  in the direction moving towards the end vertices of  $\Gamma'$ . Then the following two cases occur:

(1) Either  $\tilde{D}_i = \tilde{I}_i$ . In this case  $\tilde{D}_j = \tilde{D}_i$ .

(2) Or  $\tilde{I}_i \subsetneq \tilde{D}_i$ . In this case  $\tilde{D}_j = \tilde{I}_i$  and we have a natural exact sequence

$$1 \rightarrow \tilde{D}_j \rightarrow \tilde{D}_i \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow 0.$$

Furthermore, in the case (2) if  $\tilde{X}_i$  denotes the image of  $X_i$  in the quotient  $\tilde{\mathcal{X}}'/\tilde{I}_i$  of  $\tilde{\mathcal{X}}'$  by  $\tilde{I}_i$  then the natural morphism  $\tilde{X}_i \rightarrow P_i$ , where  $P_i \xrightarrow{\sim} \mathbb{P}_k^1$  is the image of  $X_i$  in  $\Gamma''$ , is a Galois cover of degree  $p$  ramified above a unique point  $\infty \in P_i$  [which is the edge of the geodesic linking  $P_i$  to  $P_0$ , which is linked to  $P_i$ ] with Hasse conductor  $m = 1$  at  $\infty$ .

In particular, when we move in the graph  $\Gamma'$  starting from  $X_0$  towards the end vertices of  $\Gamma'$  then the cardinality of the decomposition group  $\tilde{D}_i$  (resp. the cardinality of the inertia subgroup  $\tilde{I}_i$ ) of a vertex  $X_i$  decreases. More precisely, if when moving from a vertex  $X_i$  towards the end vertices of  $\Gamma'$  we encounter a vertex  $X_j$  then  $\tilde{D}_j \subseteq \tilde{D}_i$  and  $\tilde{I}_j \subseteq \tilde{I}_i$ .

(iv) Let  $X_i$  be a separable vertex of  $\Gamma'$  [i.e.  $\tilde{I}_i = \{1\}$ ] which is distinct from  $X_0$ . Then either  $X_i$  is an internal vertex [of  $\Gamma'$ ] which is adjacent to an end vertex of the graph  $\Gamma'$ . Furthermore,  $\tilde{D}_i = \mathbb{Z}/p\mathbb{Z}$  in this case and  $X_i$  is a Galois cover of  $\mathbb{P}_k^1$  ramified above a unique point  $\infty \in \mathbb{P}_k^1$  with Hasse conductor  $m = 1$  at  $\infty$ . [In this case if  $X_j$  is the end vertex of  $\Gamma'$  which is adjacent to  $X_i$  then  $\tilde{D}_j = \{1\}$  (cf. (ii), (2))]. Or,  $X_i$  is an end vertex of  $\Gamma'$ , and two cases can occur: either  $\tilde{D}_i = \mathbb{Z}/p\mathbb{Z}$  and  $X_i$  is a Galois cover of  $\mathbb{P}_k^1$  ramified above a unique point  $\infty \in \mathbb{P}_k^1$  [which is the point linking  $X_i$  to the rest of the tree  $\Gamma'$ ] with Hasse conductor  $m = 1$  at  $\infty$ , or  $\tilde{D}_i = \{1\}$  and  $X_i$  is adjacent to a [unique] internal separable vertex  $X_j$  with

$\tilde{D}_j \xrightarrow{\sim} \mathbb{Z}/p\mathbb{Z}$ ,  $\tilde{I}_j = \{1\}$ , and  $X_j$  is a Galois cover of  $\mathbb{P}_k^1$  ramified above a unique point  $\infty \in \mathbb{P}_k^1$  [which is the edge of the geodesic linking  $P_i$  to  $P_0$ , which is linked to  $P_i$ ] with Hasse conductor  $m = 1$  at  $\infty$ .

Let  $0 < j \leq n$  be an integer. Let  $x \in \tilde{\mathcal{X}}_K$  be a ramified point in the morphism  $f_K : \tilde{\mathcal{X}}_K \rightarrow \tilde{\mathcal{Y}}_K$ . We say that the ramified point  $x$  is of type  $j$  if the inertia subgroup  $\tilde{I}_x \subseteq G$  at  $x$  is isomorphic to  $\mathbb{Z}/p^j\mathbb{Z}$ . A vertex [irreducible component]  $X_i$  of  $\Gamma'$  is called a ramified vertex of type  $j$  if there exists a ramified point  $x$  of type  $j$  in the morphism  $f_K : \tilde{\mathcal{X}}_K \rightarrow \tilde{\mathcal{Y}}_K$  which specialises in the component  $X_i$ .

(v) Let  $X_i$  be a ramified component of  $\Gamma'$ . Then  $X_i$  is of type  $j$  for a unique integer  $0 < j \leq n$ . In other words if  $0 < j < j' \leq n$  are integers then ramified points  $x \in \tilde{\mathcal{X}}_K$  (resp.  $x' \in \tilde{\mathcal{X}}_K$ ) of type  $j$  (resp. type  $j'$ ) in the morphism  $g_K : \mathcal{X}_K \rightarrow \mathbb{P}_K^1$  specialise in distinct irreducible components of  $\tilde{\mathcal{X}}_k$ . More precisely, if  $X_i$  is a ramified vertex of type  $j$  then the inertia subgroup  $\tilde{I}_i$  which is associated to  $X_i$  has cardinality  $p^j$ , i.e.  $I_i \xrightarrow{\sim} \mathbb{Z}/p^j\mathbb{Z}$ . [In other words the type  $j$  of a ramified component  $X_i$  is uniquely determined by  $X_i$ ]. Furthermore, let  $Y_i$  be the image of  $X_i$  in  $\Gamma''$ . Then the natural morphism  $X_i \rightarrow Y_i$  has the structure of a  $\mu_{p^j}$ -torsor outside the specialisation of the branched points in  $Y_i$ , and the double points of  $\tilde{\mathcal{Y}}'_k$  which are supported by  $Y_i$ .

(vi) Let  $X_i$  be a ramified vertex of  $\tilde{\mathcal{X}}'_k$  of type  $j$ . Then when moving in the graph  $\Gamma'$  from  $X_i$  towards the end vertices of  $\Gamma'$  we encounter at most a unique ramified vertex  $X_{i'} \neq X_i$ . Moreover, in such a component  $X_{i'}$  specialises a unique ramified point in the morphism  $f_K : \tilde{\mathcal{X}}'_K \rightarrow \tilde{\mathcal{Y}}'_K$ , and the component  $X_{i'}$  is necessarily of the same type  $j$  as  $X_i$ . [In other words the graph  $\Gamma'$  separates the directions of the ramified components of  $\Gamma'$  which are of distinct types].

(vii) Assume that  $\tilde{\mathcal{X}}'$  is minimal [with respect to its defining properties above]. Then the ramified vertices in the graph  $\Gamma'$  are the end vertices of the tree  $\Gamma'$ .

*Proof.* Similar to the proof of Proposition 3.5.1.  $\square$

Our main result in this section is the following, which describes the semi-stable reduction of fake liftings of cyclic Galois covers between smooth curves [assuming they exist], and shows that fake liftings [if they exist] have semi-stable models with some very specific properties which in some sense are reminiscent to the properties of semi-stable models of smooth liftings of cyclic Galois covers between curves (cf. Proposition 3.5.1).

**Theorem 3.5.4.** *Let  $f_k : Y_k \rightarrow \mathbb{P}_k^1$  be a finite ramified Galois cover with Galois group  $G \xrightarrow{\sim} \mathbb{Z}/p^n\mathbb{Z}$  a cyclic group of order  $p^n$ ,  $n \geq 1$ , with  $Y_k$  a smooth  $k$ -curve. Let  $g_k : X_k \rightarrow \mathbb{P}_k^1$  be the [unique] cyclic sub-cover of  $f_k$  with Galois group  $H \xrightarrow{\sim} \mathbb{Z}/p^{n-1}\mathbb{Z}$  of cardinality  $p^{n-1}$ . Assume that there exists a smooth Galois lifting  $g : \mathcal{X} \rightarrow \mathbb{P}_R^1$  of  $g_k$  defined over  $R$  (cf. Definition 2.5.2), and that  $f_k$  satisfies the assumption (A) [with respect to the smooth lifting  $g$  of  $g_k$ ] (cf. 3.3.1). Let  $\tilde{f} : \mathcal{Y} \rightarrow \mathbb{P}_R^1$  be a fake lifting [relative to the smooth lifting  $g$  of  $g_k$ ] of the Galois cover  $f_k : Y_k \rightarrow \mathbb{P}_k^1$  which dominates the smooth lifting  $g$  of  $g_k$ , and which we suppose defined over  $R$  (cf. Definition 3.3.2).*

Assume that there exists a minimal birational morphism  $\mathcal{Y}' \rightarrow \mathcal{Y}$  with  $\mathcal{Y}'_k \stackrel{\text{def}}{=} \mathcal{Y}' \times_R k$  semi-stable, and such that the ramified points in the morphism  $f_K : \mathcal{Y}_K \rightarrow \mathbb{P}_K^1$  specialise in smooth distinct points of  $\mathcal{Y}'_k$ . Let  $\Gamma$  be the graph associated to the semi-stable  $k$ -curve  $\mathcal{Y}'_k$ . Write  $Y_0$  for the [irreducible component] vertex of  $\Gamma$  which

is the strict transform of  $\mathcal{Y}_k$  [ $\mathcal{Y}_k$  is irreducible] in  $\mathcal{Y}'_k$ . For a vertex  $Y_i$  of  $\Gamma$  we denote by  $D_i$  (resp.  $I_i \subseteq D_i$ ) the decomposition (resp. inertia) subgroup of  $G$  at the generic point of  $Y_i$ . Then the followings hold.

(i) The graph  $\Gamma$  is a tree.

(ii) The vertex  $Y_0 \in \text{Ver}(\Gamma)$  is a separable vertex [i.e.  $I_0 = \{1\}$ ], and  $D_0 = G$ .

Let  $H' \xrightarrow{\sim} \mathbb{Z}/p\mathbb{Z}$  be the unique subgroup of  $G$  with cardinality  $p$ . Let  $\mathcal{X}' \stackrel{\text{def}}{=} \mathcal{Y}'/H'$ , and  $\mathcal{P} \stackrel{\text{def}}{=} \mathcal{Y}'/G$ . Then  $\mathcal{X}'$  and  $\mathcal{P}$  are semi-stable  $R$ -curves, and we have a commutative diagram where the vertical maps are birational morphisms:

$$\begin{array}{ccccc} \mathcal{Y} & \xrightarrow{h} & \mathcal{X} & \xrightarrow{g} & \mathbb{P}_R^1 \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{Y}' & \xrightarrow{\tilde{h}} & \mathcal{X}' & \xrightarrow{\tilde{g}} & \mathcal{P} \end{array}$$

Let  $\Gamma'$  (resp.  $\Gamma''$ ) be the graph associated to the semi-stable  $k$ -curve  $\mathcal{X}'_k$  (resp.  $\mathcal{P}_k$ ). Then the graphs  $\Gamma'$  and  $\Gamma''$  are trees (cf. Proposition 3.5.1, (i)), and we have natural morphisms of graphs [actually these are morphisms of trees by Proposition 3.5.1 (i), and (i) above]

$$\Gamma \rightarrow \Gamma' \rightarrow \Gamma''.$$

Let  $Y_i$  be a vertex of  $\Gamma$  which is distinct from  $Y_0$ . Let  $X_i$  (resp.  $P_i$ ) be the image of  $Y_i$  in  $\mathcal{X}'$  (resp.  $\mathcal{P}$ ). Let  $\tilde{D}_i$  (resp.  $\tilde{I}_i \subseteq \tilde{D}_i$ ) be the decomposition subgroup (resp. inertia subgroup) of the Galois group  $H \stackrel{\text{def}}{=} G/H'$  which is associated to the generic point of the irreducible component  $X_i$ .

(iii) We have a natural exact sequence

$$0 \rightarrow H' \rightarrow D_i \rightarrow \tilde{D}_i \rightarrow 0.$$

Furthermore, either we have an exact sequence

$$0 \rightarrow H' \rightarrow I_i \rightarrow \tilde{I}_i \rightarrow 0.$$

In particular,  $H' \subseteq I_i$  in this case. Or

$$I_i = \tilde{I}_i = \{1\},$$

and the inertia subgroups  $I_i$  and  $\tilde{I}_i$  are trivial. The later case can occur only if  $X_i$  is adjacent, or equal, to an end vertex of  $\Gamma'$  (cf. Proposition 3.5.1, (iv)). [See (v) below for a more precise statement related to this case].

Let  $0 < j \leq n$  be an integer. Let  $y \in \mathcal{Y}_K$  be a ramified point in the morphism  $f_K : \mathcal{Y}_K \rightarrow \mathbb{P}_K^1$ . We say that the ramified point  $y$  is of type  $j$  if the inertia subgroup  $I_y \subseteq G$  at  $y$  is isomorphic to  $\mathbb{Z}/p^j\mathbb{Z}$ . A vertex [irreducible component]  $Y_i$  of  $\Gamma$  is called a ramified vertex of type  $j$  if there exists a ramified point  $y$  of type  $j$  in the morphism  $\tilde{f}_K : \mathcal{Y}_K \rightarrow \mathbb{P}_K^1$  which specialises in the component  $Y_i$ . The followings hold.

(iv) Let  $Y_i$  be a ramified vertex of  $\Gamma$ . Then  $Y_i$  is of type  $j$  for a unique integer  $0 < j \leq n$ . In other words if  $0 < j < j' \leq n$  are integers then ramified points  $y \in \mathcal{Y}_K$  (resp.  $y' \in \mathcal{Y}_K$ ) of type  $j$  (resp. type  $j'$ ) in the morphism  $f_K : \mathcal{Y}_K \rightarrow \mathbb{P}_K^1$  specialise in distinct irreducible components of  $\mathcal{Y}_k$ . Furthermore,  $D_i = I_i \xrightarrow{\sim} \mathbb{Z}/p^j\mathbb{Z}$

in this case, and the natural morphism  $Y_i \rightarrow P_i$  has the structure of a  $\mu_{p^j}$ -torsor outside the specialisation of the branched points in  $P_i$  and the double points of  $\mathcal{P}_k$  which are supported by  $P_i$ .

(v) The set of separable vertices of  $\Gamma$  which are distinct from  $Y_0$  is non empty. Furthermore, let  $Y_i$  be a separable vertex of  $\Gamma$  [i.e.  $I_i = \{1\}$  is trivial] which is distinct from  $Y_0$ . Then  $Y_i$  is an end vertex of  $\Gamma$ , and either  $D_i \xrightarrow{\sim} \mathbb{Z}/p\mathbb{Z}$  or  $D_i \xrightarrow{\sim} \mathbb{Z}/p^2\mathbb{Z}$ . [In other words the cardinality of  $D_i$  is  $\leq p^2$ ]. In the second case the natural morphism  $Y_i \rightarrow P_i$  is Galois with group  $D_i \xrightarrow{\sim} \mathbb{Z}/p^2\mathbb{Z}$ ,  $X_i \rightarrow P_i$  is its unique Galois sub-cover of degree  $p$ , and  $X_i$  is ramified above a unique point  $\infty$  of  $P_i$  with Hasse conductor 1 at  $\infty$ . [In particular,  $X_i \xrightarrow{\sim} \mathbb{P}_k^1$  in this case]. Moreover, the genus of  $Y_i$  is  $> 0$ . Moreover, no separable vertex of  $\Gamma$  is a ramified vertex.

(vi) When we move in the tree  $\Gamma$  from a given vertex towards the end vertices of  $\Gamma$  we encounter either ramified vertices or separable vertices of  $> 0$  genus [the later are necessarily end components by (v) above]. In particular, an end vertex of the graph  $\Gamma$  [which is a tree by (i)] is either a ramified vertex or a separable vertex of  $\Gamma$ .

*Proof.* The assertion in (ii) is clear since the natural morphism  $Y_0 \rightarrow \mathbb{P}_k^1$  is generically Galois with Galois group  $G$ .

Next, we will prove the assertion (iii). Let  $Y_i$  be a vertex of  $\Gamma$  which is distinct from  $Y_0$ . Let  $X_i$  (resp.  $P_i$ ) be the image of  $Y_i$  in  $\mathcal{X}'$  (resp.  $\mathcal{P}$ ). Let  $\tilde{D}_i$  (resp.  $\tilde{I}_i$ ) be the decomposition subgroup (resp. inertia subgroup) of the Galois group  $H \stackrel{\text{def}}{=} G/H'$  which is associated to the generic point of the irreducible component  $X_i$ .

The image of the decomposition group  $D_i$  in  $G/H$  via the natural morphism  $G \twoheadrightarrow G/H$  coincides with  $\tilde{D}_i$ . Hence we necessarily either have an exact sequence  $0 \rightarrow H' \rightarrow D_i \rightarrow \tilde{D}_i \rightarrow 0$ , since the group  $G$  is cyclic, or we have  $D_i = \tilde{D}_i = \{1\}$  [if  $D_i \cap H' = \{1\}$  then  $D_i = \{1\}$  is trivial] in which case the vertex  $X_i$  (resp.  $Y_i$ ) is an end vertex of  $\Gamma'$  (resp. of  $\Gamma$ ) (cf. Proposition 3.5.1, (iv)). The later case can not occur for otherwise the irreducible component  $Y_i$  would be a projective line which is an end vertex of  $\Gamma$ , and is not a ramified vertex of  $\Gamma$  [as is easily seen since  $I_i = \tilde{I}_i = \{1\}$  (cf. [Sa], Proposition 4.1.1)], hence can be contracted in the semi-stable model  $\mathcal{Y}'$  without destroying the defining properties of  $\mathcal{Y}'$ , and this would contradict the minimal character of  $\mathcal{Y}'$ . Also the image of the subgroup  $I_i$  in  $G/H$  via the natural morphism  $G \twoheadrightarrow G/H$  coincides with  $\tilde{I}_i$ . Hence we either have an exact sequence  $0 \rightarrow H' \rightarrow I_i \rightarrow \tilde{I}_i \rightarrow 0$ , or the inertia groups  $I_i = \tilde{I}_i = \{1\}$  are trivial, since the group  $G$  is cyclic. The later case can occur only if  $X_i$  is adjacent, or equal, to an end vertex of  $\Gamma'$  (cf. Proposition 3.5.1, (iv)).

Next, we prove the first assertion in (v). Assume that the set of separable vertices of  $\Gamma$  which are distinct from  $Y_0$  is empty. Let  $Y_i$  be a vertex of  $\Gamma$  which is distinct from  $Y_0$ , and  $X_i$  its image in  $\Gamma'$ . The inertia subgroup  $I_i \neq \{1\}$  is non trivial by assumption and we have a natural exact sequence  $0 \rightarrow H' \rightarrow I_i \rightarrow \tilde{I}_i \rightarrow 0$  (cf. (iii)). In particular, the natural morphism  $Y_i \rightarrow X_i$  is radicial hence a homeomorphism. Thus,  $Y_i$  is a projective line. Moreover, the natural morphism of graphs  $\Gamma \rightarrow \Gamma'$  is a homeomorphism in this case, and the graph  $\Gamma$  is a tree. In particular, the arithmetic genus of the special fibre  $\mathcal{Y}'_k$  is equal to the genus of  $Y_k$ . Hence the genera of  $Y_K$  and  $Y_k$  are equal. This implies that  $Y_K$  has good reduction, which contradicts the fact that  $\mathcal{Y}$  is a fake lifting of  $f_k$  [more precisely this contradicts the fact that  $\mathcal{Y}$  is



not smooth over  $R$  (cf. Definition 3.3.2)].

Next, we prove the assertion (i). In the course of proving (i) we will also prove the second assertion in (v). Let's move in the graph  $\Gamma'$  starting from the origin vertex  $X_0$  towards a given end vertex  $X_{\tilde{i}}$  [of  $\Gamma'$ ] along the geodesic  $\gamma$  of  $\Gamma'$  which links  $X_0$  and  $X_{\tilde{i}}$ . Let  $X_i$  be a vertex of  $\gamma$  which is distinct from both  $X_0$  and  $X_{\tilde{i}}$ . Then  $X_i$  is an internal vertex of  $\Gamma'$ , and the pre-image of  $X_i$  in  $\Gamma$  via the natural morphism  $\Gamma \rightarrow \Gamma'$  consists of a unique vertex  $Y_i$  (cf. (iii) above, more precisely the exact sequence  $0 \rightarrow H' \rightarrow D_i \rightarrow \tilde{D}_i \rightarrow 0$ ). Moreover, the natural morphism  $Y_i \rightarrow X_i$  is either radicial [this occurs only if  $H' \subseteq I_i$ ], or is a separable morphism in which case  $I_i = \tilde{I}_i = \{1\}$ , and  $X_i$  is adjacent to an end vertex of  $\Gamma'$  as follows from (iii). In fact we will show below that the later case can not occur. Let now  $Y_{\tilde{i}}$  be the unique vertex of  $\Gamma$  which is in the pre-image of the end vertex  $X_{\tilde{i}}$  of  $\Gamma'$ . The following two cases occur. Either the inertia subgroup  $I_{\tilde{i}} \neq \{1\}$  [of the group  $G$ ] which is associated to the vertex  $Y_{\tilde{i}}$  is non trivial, in which case we have an exact sequence  $0 \rightarrow H' \rightarrow I_{\tilde{i}} \rightarrow \tilde{I}_{\tilde{i}} \rightarrow 0$ , or the inertia subgroups  $I_{\tilde{i}} = \tilde{I}_{\tilde{i}} = \{1\}$  are trivial. In the first case the natural morphism  $Y_{\tilde{i}} \rightarrow X_{\tilde{i}}$  is radicial, hence a homeomorphism.

In summary two cases occur: either for every vertex  $X_i$  of the geodesic  $\gamma$  which is distinct from  $X_0$  [in particular  $X_i$  may be equal to  $X_{\tilde{i}}$ ] and its unique pre-image  $Y_i$  in  $\Gamma$  we have  $I_i \neq \{1\}$  [in particular,  $H \subseteq I_i$  in this case], or there exists a vertex  $X_i$  of  $\gamma$  which is distinct from  $X_0$  and its unique pre-image  $Y_i$  in  $\Gamma$  such that  $I_i = \tilde{I}_i = \{1\}$ .

In the first case the natural morphism  $Y_i \rightarrow X_i$  is radicial and the natural morphism  $\tilde{h}^{-1}(\gamma) \rightarrow \gamma$ , where  $\tilde{h}^{-1}(\gamma)$  is the pre-image of  $\gamma$  in  $\Gamma$ , is a homeomorphism. In particular,  $\tilde{h}^{-1}(\gamma)$  is a tree in this case. More precisely, in this case  $\tilde{h}^{-1}(\gamma)$  is a geodesic which links  $Y_0$  to the unique vertex  $Y_{\tilde{i}}$  in the pre-image of  $X_{\tilde{i}}$  which is an end vertex of  $\Gamma$ . Moreover, all vertices of  $\tilde{h}^{-1}(\gamma)$  which are distinct from  $X_0$  are projective lines in this case and the vertex  $Y_{\tilde{i}}$  is necessarily a ramified vertex. For otherwise the component  $Y_{\tilde{i}}$  would be a [non ramified] projective line hence can be contracted in the semi-stable model  $\mathcal{Y}'$  without destroying the defining properties of  $\mathcal{Y}'$ , and this would contradict the minimal character of  $\mathcal{Y}'$ . Now we shall investigate the second case.

Assume that the second case above occurs. In order to show that the graph  $\Gamma$  is a tree it suffices to show that the pre-image  $\tilde{h}^{-1}(\gamma)$  of the geodesic  $\gamma$  is also a tree in this case [for every possible choice of  $\gamma$ ]. More precisely, we will show that the natural map  $\tilde{h}^{-1}(\gamma) \rightarrow \gamma$  is a homeomorphism of trees. Let  $X_i$  be the first vertex of  $\gamma$  that we encounter when moving from  $X_0$  towards  $X_{\tilde{i}}$ , and  $Y_i$  the unique pre-image of  $X_i$  in  $\Gamma$ , such that the inertia groups  $I_i = \tilde{I}_i = \{1\}$  are trivial. We will show that  $X_i = X_{\tilde{i}}$  is necessarily the end vertex of  $\gamma$  and that the natural morphism  $Y_i \rightarrow X_i$ , which is generically Galois [with Galois group  $H'$ ], is only [totally] ramified above the unique double point  $x_i$  of  $\mathcal{X}'_k$  which is supported by  $X_i$ . This will complete the proof of the assertion that  $\Gamma$  is a tree, and will also prove the second assertion in (v).

Assume the contrary that  $X_i \neq X_{\tilde{i}}$  is not the end vertex of  $\gamma$ . Then  $X_i$  is an internal vertex of  $\Gamma$ , which is linked to a unique double point  $x_i$  which is an edge of the geodesic which links  $X_i$  to  $X_0$ , and is linked to [at least] another double point  $x_{i'}$  which is an edge of the geodesic which links  $X_i$  to  $X_{\tilde{i}}$  [there may be more double points linked to  $X_i$  which are edges of the possible geodesics linking  $X_i$  to other

end vertices of  $\Gamma'$ . Moreover,  $\tilde{D}_i \xrightarrow{\sim} \mathbb{Z}/p\mathbb{Z}$  in this case (cf. Proposition 3.5.1, (iv)) which necessarily implies that  $D_i \xrightarrow{\sim} \mathbb{Z}/p^2\mathbb{Z}$ , and the natural morphism  $X_i \rightarrow P_i$  [where  $P_i$  is the image of  $X_i$  in  $\mathcal{P}$ ] is a Galois cover of degree  $p$  ramified above a unique point  $\infty \in P_i$  [which is the image of the double point  $x_i$  in  $\mathcal{P}$ ] with Hasse conductor  $m = 1$  at  $\infty$  (cf. Proposition 3.5.1 (iii)). [In particular,  $X_i \xrightarrow{\sim} \mathbb{P}_k^1$  is a projective line].

The natural morphism  $Y_i \rightarrow X_i$  is a generically Galois morphism with Galois group  $\mathbb{Z}/p\mathbb{Z}$ , and is ramified above the double point  $x_i$  with Hasse conductor  $m_i$  at this point [if  $X_j$  is the vertex of  $\gamma$  such that  $x_i = X_i \cap X_j$  and  $Y_j$  its unique pre-image in  $\Gamma$  then  $I_j \neq \{1\}$  by assumption]. Above the double point  $x_{i'}$  this morphism is either ramified with Hasse conductor  $m_{i'}$  or is unramified. In both cases the double point  $x_{i'}$  produces a non trivial contribution to the arithmetic genus of  $\mathcal{Y}'_k$ . More precisely, in the first case the contribution of  $x_{i'}$  to the arithmetic genus is  $p - 1$ , and in the second case it is  $\frac{(m_{i'}+1)(p-1)}{2}$ .

We will construct, in order to contradict the above assumption, a new Garuti lifting  $f_1 : \mathcal{Y}_1 \rightarrow \mathbb{P}_R^1$  of the Galois cover  $f_k : Y_k \rightarrow \mathbb{P}_k^1$  which dominates the smooth lifting  $g : \mathcal{X} \rightarrow \mathbb{P}_R^1$  of the Galois subcover  $g_k : X_k \rightarrow \mathbb{P}_R^1$  of degree  $p^{n-1}$ , and such that the degree of ramification  $\delta_1 \stackrel{\text{def}}{=} \delta_{f_{1,K}}$  in the morphism  $f_{1,K} : \mathcal{Y}_{1,K} \rightarrow \mathbb{P}_K^1$  between generic fibres satisfies the inequality  $\delta_1 < \delta \stackrel{\text{def}}{=} \delta_{\tilde{f}_K}$ . This would contradict the minimality of  $\delta$ , i.e. contradicts the fact that  $\tilde{f}$  is a fake lifting of  $f_k$ . To simplify the arguments below we will assume that  $G = D_i = \mathbb{Z}/p^2\mathbb{Z}$ . [The construction of  $f_1$  in the general case is done in a similar fashion by using induced covers from  $D_i$  to  $G$  (cf. the construction of Garuti in [Ga], 3, for similar arguments)].

Let  $X_{1,k}$  be the semi-stable  $k$ -curve which is obtained from  $\mathcal{X}'_k$  by removing the geodesic of the graph  $\Gamma'$  which links  $X_i$  to the terminal vertex  $X_{\tilde{i}}$ , with the vertex  $X_i$  removed. Thus,  $X_{1,k}$  is a semi-stable  $k$ -curve with the same arithmetic genus as  $\mathcal{X}'_k$  [which is the same as that of  $X_k$ ]. Moreover, the graph associated to the semi-stable  $k$ -curve  $X_{1,k}$  is a tree with origin vertex  $X_0$ , and the irreducible component  $X_i$  is an end vertex of this tree. Let  $P_{1,k}$  be the image of  $X_{1,k}$  in  $\mathcal{P}$  [here we view  $X_{1,k}$  as a closed sub-scheme of  $\mathcal{X}_k$ ], and  $Y_{1,k}$  the pre-image of  $X_{1,k}$  in  $\mathcal{Y}'_k$ . We have natural finite morphisms  $Y_{1,k} \rightarrow X_{1,k} \rightarrow P_{1,k}$  between semi-stable  $k$ -curves.

One can construct a new finite morphism  $Y'_{1,k} \rightarrow X_{1,k} \rightarrow P_{1,k}$  which above  $P_{1,k} \setminus P_i$  coincides with the finite cover which is induced by the above cover  $Y_{1,k} \rightarrow X_{1,k} \rightarrow P_{1,k}$ , above  $P_i$  is a generically separable Galois cover with Galois group  $D_i = G$  which is ramified only above the unique double point  $\infty$  of  $P_{1,k}$  linking  $P_i$  to the geodesic of  $\Gamma''$  which links  $P_i$  and  $P_0$  [the point  $\infty$  is the image of  $x_i$  in  $\mathcal{P}$ ], and which above the formal completion of  $P_{1,k}$  at the double point  $\infty$  coincides with the cover that is induced by the morphisms  $Y_{1,k} \rightarrow X_{1,k} \rightarrow P_{1,k}$ . In other words in this new cover we eliminate all the irreducible components of the geodesic  $\gamma$  that we encounter when moving from  $X_i$  in the direction of  $X_{\tilde{i}}$ , and we also eliminate the ramification in the morphism  $Y_i \rightarrow X_i$  which may arise above points of  $X_i$  which are distinct from the double point  $x_i$  (cf. discussion above).

The finite morphisms  $Y'_{1,k} \rightarrow X_{1,k} \rightarrow P_{1,k}$  can be lifted [uniquely] to finite morphisms  $\tilde{\mathcal{Y}}_1 \rightarrow \mathcal{X}_1 \rightarrow \mathcal{P}_1$ , where  $\tilde{\mathcal{Y}}_1 \rightarrow \mathcal{P}_1$  is a Galois cover with Galois group  $G$  which lifts the finite morphism  $Y'_{1,k} \rightarrow P_{1,k}$ , and  $\mathcal{X}_1 \rightarrow \mathcal{P}_1$  is the unique sub-cover with Galois group  $H$  which lifts the finite morphism  $X_{1,k} \rightarrow P_{1,k}$ , as follows.

First, we have a natural Galois lifting of the finite morphism  $Y'_{1,k} \setminus Y'_1 \rightarrow P_{1,k} \setminus P_1$

which is the restriction of the finite Galois morphism  $\mathcal{Y}' \rightarrow \mathcal{P}$  to the formal fibre of  $P_{1,k} \setminus P_1$  in  $\mathcal{P}$ . The restriction of the finite morphism  $\mathcal{Y}' \rightarrow \mathcal{P}$  to the formal fibre at the double point  $\infty$  [above] provides a natural lifting of the cover above the formal completion of  $P_{1,k}$  at the double point  $\infty$  which is induced by  $Y_{1,k} \rightarrow X_{1,k} \rightarrow P_{1,k}$ . Second, the restriction of the finite morphism  $Y'_{1,k} \rightarrow X_{1,k} \rightarrow P_{1,k}$  to the irreducible component  $P_1 \setminus \{\infty\}$  [which is an étale torsor] can be lifted to an étale torsor of the formal fibre of  $P_1 \setminus \{\infty\}$  in  $\mathcal{P}_1$  with Galois group  $G$  by the theorems of liftings of étale covers (cf. [Gr]). These liftings can be patched using formal patching techniques to construct the required Galois cover  $\tilde{\mathcal{Y}}_1 \rightarrow \mathcal{X}_1 \rightarrow \mathcal{P}_1$  (cf. [Ga], and Proposition 1.2.2).

Let's now contract [in a Galois equivariant fashion] in  $\tilde{\mathcal{Y}}_1$  (resp. in  $\mathcal{X}_1$ ) all the irreducible components of the special fibre  $\tilde{\mathcal{Y}}_{1,k}$  (resp.  $\mathcal{X}_{1,k}$ ) which are distinct from  $Y_0$  (resp. distinct from  $X_0$ ). We then obtain a normal  $R$ -curve  $\mathcal{Y}_1$  (resp. obtain the smooth  $R$ -curve  $\mathcal{X}$ ). We have natural finite Galois morphisms  $f_1 : \mathcal{Y}_1 \rightarrow \mathcal{X} \xrightarrow{g} \mathbb{P}_R^1$ , and the Galois cover  $f_1 : \mathcal{Y}_1 \rightarrow \mathbb{P}_R^1$  is by construction a Garuti lifting of the Galois cover  $f_k : Y_k \rightarrow \mathbb{P}_k^1$  [the fact that  $f_1$  dominates the smooth lifting  $g : \mathcal{X} \rightarrow \mathbb{P}_R^1$  of  $g_k$  is easily verified, and follows from the above construction]. Let  $\delta_1 \stackrel{\text{def}}{=} \delta_{f_1,K}$  be the degree of the different in the cover  $f_{1,K} : \mathcal{Y}_{1,K} \rightarrow \mathbb{P}_K^1$  between generic fibres. Then clearly [by construction] we have  $\delta_1 < \delta$ , since the only point of the irreducible component  $X_i$  of  $\mathcal{X}_{1,k}$  which contributes to the arithmetic genus of  $\mathcal{Y}_{1,k}$  is the double point  $x_i$  [and this contribution is the same contribution as in the original cover  $\mathcal{Y}'_k \rightarrow \mathcal{P}_k$  by construction] (cf. the discussion above). But this contradicts the minimality of  $\delta$ , and the fact that  $\tilde{f} : \mathcal{Y} \rightarrow \mathbb{P}_R^1$  is a fake lifting of the Galois cover  $f_k$ .

This shows that the irreducible component  $X_i = X_{\tilde{i}}$  is necessarily an end vertex of the geodesic  $\gamma$  [hence also an end vertex of the graph  $\Gamma'$ ]. A similar argument shows that the natural morphism  $Y_i \rightarrow X_i$  [which is generically separable] is only ramified above the unique double point  $x_i$  of  $X_i$ . This, in particular, shows that  $\tilde{h}^{-1}(\gamma)$  is a tree, and the natural morphism  $h^{-1}(\gamma) \rightarrow \gamma$  is a homeomorphism of trees. Thus, the graph  $\Gamma$  is a tree as claimed. Furthermore,  $Y_i$  can not be a ramified component by [Sa], Corollary 4.1.2, which proves the last assertion in (v).

The proof of (iv) is similar to the proof of Proposition 3.5.1 (v).

The proof of the second assertion in (v) follows from Proposition 3.5.1, (iv), and [Sa], Corollary 4.1.2.

Finally, we prove (vi). Let  $Y_i$  be an internal vertex of  $\Gamma$ , and  $Y_{\tilde{i}}$  an end vertex of  $\Gamma$  which we encounter when moving from  $Y_i$  towards the end vertices. Let  $\gamma$  be the geodesic of  $\Gamma$  which links  $Y_i$  and  $Y_{\tilde{i}}$ . Assume that  $Y_{\tilde{i}}$  is neither a ramified component nor a separable component. Then all vertices of  $\gamma$  are projective lines as is easily seen, and can be contracted in  $\mathcal{Y}'$  without destroying the defining properties of  $\mathcal{Y}'$ , which would contradict the minimal character of  $\mathcal{Y}'$ . Thus,  $Y_i$  is a terminal vertex as claimed.  $\square$

The following Lemma 3.5.5 is used in the proof of Proposition 3.5.1 (v), and Theorem 3.5.4 (vi).

**Lemma 3.5.5.** *Let  $\mathcal{X} \stackrel{\text{def}}{=} \text{Spf } A$  be a connected smooth  $R$ -formal affine scheme. Let  $f : \mathcal{Y} \rightarrow \mathcal{X}$  be a finite Galois cover between smooth  $R$ -formal schemes with  $\mathcal{Y}$  connected, with Galois group  $G \xrightarrow{\sim} \mathbb{Z}/p^n\mathbb{Z}$ ,  $n \geq 1$ , and such that the natural morphism  $f_K : \mathcal{Y}_K \rightarrow \mathcal{X}_K$  between generic fibres is étale. [Here the generic fibres*

$\mathcal{Y}_K$  and  $\mathcal{X}_K$  denote the rigid analytic spaces associated to  $\mathcal{Y}$  and  $\mathcal{X}$  respectively (cf. [Ab])). Let  $\eta$  be the generic point of the special fibre of  $\mathcal{X}$  and  $\delta$  the degree of the different in the morphism  $f$  above  $\eta$ . Assume that  $\delta = v_K(p)(1+p+p^2+\dots+p^{n-1})$ . Then the natural morphism  $f_k : \mathcal{Y}_k \rightarrow \mathcal{X}_k \stackrel{\text{def}}{=} \text{Spec } A/\pi A$  between special fibres has the structure of a  $\mu_{p^n}$ -torsor.

*Proof.* The Galois cover  $f$  has a natural factorization

$$f : \mathcal{Y} = \mathcal{Y}_n \xrightarrow{f_{n-1}} \mathcal{Y}_{n-1} \rightarrow \dots \rightarrow \mathcal{Y}_2 \xrightarrow{f_1} \mathcal{Y}_1 \stackrel{\text{def}}{=} \mathcal{X},$$

where  $f_i : \mathcal{Y}_{i+1} \rightarrow \mathcal{Y}_i$  is a Galois cover of degree  $p$ . Let  $\delta_i$  be the degree of the different in the morphism  $f_i$  above the generic point  $\eta_i$  of  $\mathcal{Y}_i$ . Then  $\delta_i \leq v_K(p)$  (cf. [Sa], Proposition 2.3). The assumption on  $\delta$  implies that  $\delta_i = v_K(p)$ ,  $\forall i \in \{1, \dots, n-1\}$ . Hence  $f_i : \mathcal{Y}_{i+1} \rightarrow \mathcal{Y}_i$  is a torsor under the group scheme  $\mu_{p,R}$  (cf. loc. cit.). In fact this latter property is equivalent to  $\delta_i = v_K(p)$ . This implies in particular that the Galois cover  $f$  is given by an equation  $Z^{p^n} = u$  where  $u \in A$  is a unit whose image  $\bar{u}$  in  $A/\pi A$  is not a  $p$ -th power and hence has the structure of a  $\mu_{p^n,R}$ -torsor.  $\square$

**§4. The Smoothing Process.** In this section we introduce the process of smoothing of fake liftings of cyclic Galois covers between smooth curves. The idea of smoothing of fake liftings already germs in the proof of Theorem 3.5.4. The smoothing process ultimately aims to show that fake liftings as introduced in §3 do not exist. This in turn would imply the validity of the [revisited] Oort conjecture (cf. Remark 3.3.3).

We use the same notations as in §2, and §3. Especially the Notations 2.1.

**4.1.** Let  $n \geq 1$  be a positive integer. Let  $f_k : Y_k \rightarrow \mathbb{P}_k^1$  be a finite ramified Galois cover with Galois group  $G \xrightarrow{\sim} \mathbb{Z}/p^n\mathbb{Z}$ , a cyclic group of order  $p^n$ , with  $Y_k$  smooth over  $k$ . Let  $g_k : X_k \rightarrow \mathbb{P}_k^1$  be the [unique] cyclic sub-cover of  $f_k$  with Galois group  $H \xrightarrow{\sim} \mathbb{Z}/p^{n-1}\mathbb{Z}$  of cardinality  $p^{n-1}$ . Assume that there exists a smooth lifting  $g : \mathcal{X} \rightarrow \mathbb{P}_R^1$  of  $g_k$  defined over  $R$  (cf. Definition 2.5.2).

Assume that  $f_k$  satisfies the assumption **(A)** in 3.3.1 [with respect to the smooth lifting  $g$  of  $g_k$ ]. Let  $\tilde{f} : \mathcal{Y} \rightarrow \mathbb{P}_R^1$  be a fake lifting of the Galois cover  $f_k : Y_k \rightarrow \mathbb{P}_k^1$  [with respect to the smooth lifting  $g$  of  $g_k$ ], which dominates the smooth lifting  $g$  of  $g_k$ , and which we suppose defined over  $R$  (cf. Definition 3.3.2). We assume that there exists a minimal birational morphism  $\mathcal{Y}' \rightarrow \mathcal{Y}$  with  $\mathcal{Y}'_k \stackrel{\text{def}}{=} \mathcal{Y}' \times_R k$  semi-stable, and such that the ramified points in the morphism  $\tilde{f}_K : \mathcal{Y}_K \rightarrow \mathbb{P}_K^1$  specialise in smooth distinct points of  $\mathcal{Y}'_k$ . Let  $\Gamma$  be the graph associated to the semi-stable curve  $\mathcal{Y}'_k$  which is a tree by Theorem 3.5.4 (i).

Let  $H' \xrightarrow{\sim} \mathbb{Z}/p\mathbb{Z}$  be the unique subgroup of  $G$  with cardinality  $p$ . Let  $\mathcal{X}' \stackrel{\text{def}}{=} \mathcal{Y}'/H'$ , and  $\mathcal{P} \stackrel{\text{def}}{=} \mathcal{Y}'/G$ , be the quotient of  $\mathcal{Y}'$  by  $H'$ , and the quotient of  $\mathcal{Y}'$  by  $G$ , respectively. Then  $\mathcal{X}'$  and  $\mathcal{P}$  are semi-stable  $R$ -curves, and we have a natural Galois morphism  $f' : \mathcal{Y}' \rightarrow \mathcal{P}$  with Galois group  $G$ . We have a commutative diagram where the vertical maps are birational morphisms:

$$\begin{array}{ccccc} \mathcal{Y} & \xrightarrow{h} & \mathcal{X} & \xrightarrow{g} & \mathbb{P}_R^1 \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{Y}' & \xrightarrow{\tilde{h}} & \mathcal{X}' & \xrightarrow{\tilde{g}} & \mathcal{P} \end{array}$$

Let  $\Gamma'$  (resp.  $\Gamma''$ ) be the graph associated to the semi-stable  $k$ -curve  $\mathcal{X}'_k$  (resp.  $\mathcal{P}_k$ ). Then the graphs  $\Gamma'$  and  $\Gamma''$  are trees (cf. Proposition 3.5.1, (i)), and we have natural morphisms of trees

$$\Gamma \rightarrow \Gamma' \rightarrow \Gamma''.$$

Let  $Y_0$  be the origin vertex of  $\Gamma$  [which is the strict transform of  $\mathcal{Y}_k$  in  $\mathcal{Y}'$ ], and let  $P_0$  be its image in  $\Gamma''$  which is the origin vertex of  $\Gamma''$ .

**4.1.1 The semi-stable curve  $\mathcal{P}_i$  associated to an internal vertex  $P_i$ .** Let  $P_i$  be an internal vertex of  $\Gamma''$ . Let  $P_{i,k}$  be the semi-stable  $k$ -curve of arithmetic genus 0, which is obtained from the semi-stable  $k$ -curve  $\mathcal{P}_k \stackrel{\text{def}}{=} \mathcal{P} \times_R k$  by removing all the geodesics of  $\Gamma''$  which link the vertex  $P_i$  to the end vertices of  $\Gamma''$ , excluding the vertex  $P_i$ . The graph associated to the semi-stable curve  $P_{i,k}$  is a tree  $\Gamma''_i$  in which the vertex  $P_i$  is a terminal vertex. Denote by  $\infty$  the unique double point of  $P_{i,k}$  which is supported by  $P_i$ , and which links  $P_i$  to the geodesic of  $\Gamma''_i$  joining  $P_i$  and  $P_0$ .

Let  $\mathcal{P}_i$  be the semi-stable  $R$ -model of  $\mathbb{P}_R^1$  which is obtained from the semi-stable  $R$ -model  $\mathcal{P}$  by contracting all the irreducible components of  $\mathcal{P}_k \setminus P_{i,k}$  [here we view  $P_{i,k}$  as a closed sub-scheme of  $\mathcal{P}_k$ ]. Then the special fibre  $\mathcal{P}_{i,k} \stackrel{\text{def}}{=} \mathcal{P}_i \times_R k$  of  $\mathcal{P}_i$  equals  $P_{i,k}$ , and we have natural birational morphisms

$$\mathcal{P} \rightarrow \mathcal{P}_i \rightarrow \mathbb{P}_R^1.$$

Let  $\mathcal{P}'_i$  be the formal fibre of  $P_i \setminus \{\infty\}$  in  $\mathcal{P}_i$ . Then

$$\mathcal{P}'_i \xrightarrow{\sim} \text{Spf } R \langle S \rangle$$

is a formal closed disc. Let  $\mathcal{P}''_i$  be the formal fibre of  $P_{i,k} \setminus \{P_i\}$  in  $\mathcal{P}_i$ , and  $\mathcal{P}_{i,\infty}$  the formal fibre of  $\mathcal{P}_i$  at  $\infty$  which is a formal open annulus, i.e.

$$\mathcal{P}_{i,\infty} \xrightarrow{\sim} \text{Spf } \frac{R[[S, T]]}{(ST - \pi^e)},$$

for some integer  $e \geq 1$  [actually  $e$  is necessarily divisible by a suitable power of  $p$ ].

Note that the semi-stable  $R$ -curve  $\mathcal{P}_i$  is obtained by patching  $\mathcal{P}'_i$  and  $\mathcal{P}''_i$  along the open annulus  $\mathcal{P}_{i,\infty}$ .

Next, we define the important concept of a removable vertex in Definition 4.1.2, and the smoothening process in Definition 4.1.3.

**Definition 4.1.2 (Removable Vertex of  $\Gamma''$ ).** [We use the same notations and assumptions as above]. We say that  $P_i$  is a removable vertex of the tree  $\Gamma''$  if there exists a finite Galois cover

$$f'_1 : \mathcal{Y}'_1 \rightarrow \mathcal{P}_i$$

[where  $\mathcal{P}_i$  is as in 4.1.1] with Galois group  $G$ , satisfying the following three conditions.

(i) The restriction of the Galois cover  $f'_1$  to  $\mathcal{P}''_i$  (resp. to  $\mathcal{P}_{i,\infty}$ ) is isomorphic to the restriction of the Galois cover

$$f' : \mathcal{Y}' \rightarrow \mathcal{P}$$

[which is the semi-stable minimal model of the fake lifting  $\tilde{f} : \mathcal{Y} \rightarrow \mathbb{P}_R^1$  of  $f_k$ ] above  $\mathcal{P}_i''$  (resp. above  $\mathcal{P}_{i,\infty}$ ).

(ii) Let  $g'_1 : \mathcal{X}'_1 \rightarrow \mathcal{P}_i$  be the unique Galois sub-cover of  $f'_1$  of degree  $p^{n-1}$ . Then  $g'_1$  is generically [Galois] isomorphic to the Galois cover  $g : \mathcal{X} \rightarrow \mathbb{P}_R^1$  which is the given smooth lifting of  $g_k$ .

(iii) The arithmetic genera  $g$  (resp.  $g_1$ ) of the special fibres  $\mathcal{Y}'_k$  (resp.  $\mathcal{Y}'_{1,k} \stackrel{\text{def}}{=} \mathcal{Y}'_1 \times_R k$ ) satisfy the inequality

$$g_1 < g.$$

**Definition 4.1.3 (Smoothing of a Fake Lifting).** [We use the same notations and assumptions as above]. Assume that  $P_i$  is a removable vertex in the sense of Definition 4.1.2. Let  $f'_1 : \mathcal{Y}'_1 \rightarrow \mathcal{P}_i$  be the corresponding Galois cover with Galois group  $G$  (which is given by Definition 4.1.2). Let  $\mathcal{Y}_1$  be the normal  $R$ -curve which is obtained from  $\mathcal{Y}'_1$  by contracting all the irreducible components of  $\mathcal{Y}'_{1,k}$  which are distinct from  $Y_0$ . The Galois cover  $f'_1$  induces naturally a Galois cover

$$f_1 : \mathcal{Y}_1 \rightarrow \mathbb{P}_R^1$$

with Galois group  $G$  [since the above contraction procedure is Galois equivariant].

The inequality  $g_1 < g$  implies [in fact is equivalent to the fact] that the degree of the [generic] different  $\delta_1 \stackrel{\text{def}}{=} \delta_{f_1,K}$  in the natural morphism

$$f_{1,K} : \mathcal{Y}_{1,K} \stackrel{\text{def}}{=} \mathcal{Y}_1 \otimes_R K \rightarrow \mathbb{P}_K^1$$

between generic fibres satisfies the inequality

$$\delta_1 < \delta \stackrel{\text{def}}{=} \delta_{\tilde{f}_K}.$$

We call the Galois cover  $f_1 : \mathcal{Y}_1 \rightarrow \mathbb{P}_R^1$  a smoothing of the fake lifting  $\tilde{f} : \mathcal{Y} \rightarrow \mathbb{P}_R^1$ .

Note that [by property (ii) in Definition 4.1.2] the Galois cover  $f_1 : \mathcal{Y}_1 \rightarrow \mathbb{P}_R^1$  is a Garuti lifting of the Galois cover  $f_k : Y_k \rightarrow \mathbb{P}_k^1$ , which dominates the smooth lifting  $g : \mathcal{X} \rightarrow \mathbb{P}_R^1$  of the Galois sub-cover  $g_k : X_k \rightarrow \mathbb{P}_k^1$ . [This last property may be used to define the notion of a smoothing of a fake lifting independently from Definition 4.1.2]

**4.2.** The existence of a removable vertex in the tree  $\Gamma''$ , which implies [by definition] the existence of a smoothing  $f_1 : \mathcal{Y}_1 \rightarrow \mathbb{P}_R^1$  of the fake lifting  $\tilde{f} : \mathcal{Y} \rightarrow \mathbb{P}_R^1$  [more precisely, the above inequality  $\delta_1 < \delta$ ] (cf. Definition 4.1.3), contradicts the fact that  $\tilde{f}$  is a fake lifting [i.e. contradicts the minimality of the generic different  $\delta$  of  $\tilde{f}$ ], hence will prove the [revisited] Oort conjecture for the Galois cover  $f_k : Y_k \rightarrow \mathbb{P}_k^1$  [and the smooth lifting  $g$  of  $g_k$ ] (cf. Remark 3.3.3). More precisely, we have the following.

**Proposition 4.2.1.** *Let  $f_k : Y_k \rightarrow \mathbb{P}_k^1$  be a finite ramified Galois cover with Galois group  $G \xrightarrow{\sim} \mathbb{Z}/p^n\mathbb{Z}$ , and  $Y_k$  is a smooth  $k$ -curve. Let  $g_k : X_k \rightarrow \mathbb{P}_k^1$  be the Galois sub-cover of  $f_k$  with Galois group  $H \xrightarrow{\sim} \mathbb{Z}/p^{n-1}\mathbb{Z}$ . Assume that there exists a smooth Galois lifting  $g : \mathcal{X} \rightarrow \mathbb{P}_R^1$  of  $g_k$  defined over  $R$  (cf. Definition 2.5.2). Assume that  $f_k$  satisfies the assumption **(A)** in 3.3.1 [with respect to the smooth lifting  $g$  of  $g_k$ ].*

Let  $\tilde{f} : \mathcal{Y} \rightarrow \mathbb{P}_R^1$  be a fake lifting of the Galois cover  $f_k : Y_k \rightarrow \mathbb{P}_k^1$  [with respect to the smooth lifting  $g$  of  $g_k$ ], which dominates the smooth lifting  $g$  of  $g_k$ , which we suppose defined over  $R$  (cf. Definition 3.3.2). We assume that there exists a minimal birational morphism  $\mathcal{Y}' \rightarrow \mathcal{Y}$  with  $\mathcal{Y}'_k \stackrel{\text{def}}{=} \mathcal{Y}' \times_R k$  semi-stable, and such that the ramified points in the morphism  $f_K : \mathcal{Y}_K \rightarrow \mathbb{P}_K^1$  specialise in smooth distinct points of  $\mathcal{Y}'_k$ . Let  $\mathcal{P} \stackrel{\text{def}}{=} \mathcal{Y}'/G$  be the quotient of  $\mathcal{Y}'$  by  $G$ , and  $\Gamma''$  the tree which is associated to the special fibre  $\mathcal{P}_k$  of  $\mathcal{P}$ .

Under these assumptions suppose that there exists an internal vertex  $P_i$  of the tree  $\Gamma''$  which is a removable vertex of  $\Gamma''$  in the sense of Definition 4.1.2, or equivalently that there exists a smoothening  $f_1 : \mathcal{Y}_1 \rightarrow \mathbb{P}_R^1$  of the fake lifting  $\tilde{f} : \mathcal{Y} \rightarrow \mathbb{P}_R^1$  in the sense of Definition 4.1.3. Then the [revisited] Oort conjecture [**Conj-O2-Rev**] is true for the Galois cover  $f_k : Y_k \rightarrow \mathbb{P}_k^1$ , and the smooth lifting  $g$  of the Galois sub-cover  $g_k$ .

One can show that fake liftings of cyclic Galois covers between smooth curves, assuming they exist, always admit a smoothening in the case of cyclic Galois covers of degree  $p$ . This provides an alternative proof of the Oort conjecture in the case of a cyclic Galois group  $G \xrightarrow{\sim} \mathbb{Z}/p\mathbb{Z}$  of order  $p$ . This proof doesn't use the equation describing the degeneration of the Kummer equation of degree  $p$  to the Artin-Schreier equation (as in [Se-Oo-Su], and [Gr-Ma]), but rather uses the degeneration of the Kummer equation to a radicial equation (see proof of Proposition 4.2.2). More precisely, we have the following.

**Proposition 4.2.2.** *Assume that  $R$  contains a primitive  $p$ -th root of unity. Let  $f_k : Y_k \rightarrow \mathbb{P}_k^1$  be a finite ramified Galois cover with Galois group  $G \xrightarrow{\sim} \mathbb{Z}/p\mathbb{Z}$ , and  $Y_k$  is a smooth  $k$ -curve. Assume that  $f_k$  satisfies the assumption **(A)** in 3.3.1. The assumption **(A)** in this case means that  $f_k$  admits no smooth lifting, and a fake lifting means a Garuti lifting with minimal generic different. Let  $\tilde{f} : \mathcal{Y} \rightarrow \mathbb{P}_R^1$  be a fake lifting of the Galois cover  $f_k : Y_k \rightarrow \mathbb{P}_k^1$  [which we suppose defined over  $R$ ] (cf. Definition 3.3.2). We assume that there exists a minimal birational morphism  $\mathcal{Y}' \rightarrow \mathcal{Y}$  with  $\mathcal{Y}'_k \stackrel{\text{def}}{=} \mathcal{Y}' \times_R k$  semi-stable, and such that the ramified points in the morphism  $f_K : \mathcal{Y}_K \rightarrow \mathbb{P}_K^1$  specialise in smooth distinct points of  $\mathcal{Y}'_k$ . Let  $\mathcal{P} \stackrel{\text{def}}{=} \mathcal{Y}'/G$  be the quotient of  $\mathcal{Y}'$  by  $G$  [ $\mathcal{P}$  is a semi-stable  $R$ -model of  $\mathbb{P}_R^1$ ], and  $\Gamma'$  the tree which is associated to the special fibre  $\mathcal{P}_k$  of  $\mathcal{P}$ .*

*Then there exists an internal vertex  $P_i$  of the tree  $\Gamma'$  which is a removable vertex of  $\Gamma'$  in the sense of Definition 4.1.2. In particular, the [revisited] Oort conjecture is true for the Galois cover  $f_k : Y_k \rightarrow \mathbb{P}_k^1$  (cf. Proposition 4.2.1).*

*Proof.* We can assume, without loss of generality, that the morphism  $f_k$  is ramified above a unique point  $\infty$  of  $\mathbb{P}_k^1$ , i.e. work within the framework of [**Conj-O3**]. Let  $P_0$  be the origin vertex of the tree  $\Gamma'$ , and  $P_1$  the [unique] vertex of  $\Gamma'$  which is adjacent to  $P_0$ . We will show that  $P_1$  is a removable vertex of  $\Gamma'$ .

The semi-stable  $R$ -curve  $\mathcal{P}_1$  (cf. 4.1.1) in this case has a special fibre  $\mathcal{P}_{1,k}$  which consists of the two irreducible [smooth] components  $P_0$  and  $P_1$ , which meet at the unique double point  $\infty$ .

Let  $\mathcal{P}'_1 \xrightarrow{\sim} \text{Spf } R < \frac{1}{T} >$  be the formal fibre of  $P_1 \setminus \{\infty\}$  in  $\mathcal{P}_1$ ,  $\mathcal{P}_{1,\infty}$  the formal completion of  $\mathcal{P}_1$  at  $\infty$ , and  $\mathcal{P}''_1$  the formal fibre of  $\mathcal{P}_{1,k} \setminus P_1$  in  $\mathcal{P}_1$ . The natural Galois morphism  $\mathcal{Y}' \rightarrow \mathcal{P}$  restricts to Galois morphisms  $\mathcal{Y}'_1 \rightarrow \mathcal{P}''_1$ , and  $\mathcal{Y}'_y \rightarrow \mathcal{P}_{1,\infty}$ , where  $\mathcal{Y}'_y$  is the formal completion of  $\mathcal{Y}'$  at the unique double point  $y$  above  $\infty$ .

The degeneration type of the Galois cover  $\mathcal{Y}'_y \rightarrow \mathcal{P}_{1,\infty}$  on the boundary which is linked to  $\mathcal{P}'_1$  is necessarily radicial of type  $(\alpha_p, -m, 0)$  where  $m > 0$  is an integer prime to  $p$  [since  $P_1$  is an internal vertex of  $\Gamma'$ ], or of type  $(\mu_p, -m, 0)$  where  $m$  is as above. We only treat the first case, the second case is treated in a similar way (cf. [Sa], Proposition 3.3.1, (a2)).

In the first case the Galois cover  $\mathcal{Y}'_y \rightarrow \mathcal{P}_{1,\infty}$  induces a Galois cover on the boundary which is linked to  $\mathcal{P}'_1$  given by an equation  $X^p = 1 + \pi^{tp}T^m$ , for a suitable choice of  $T$  as above, and  $t < v_K(\lambda)$  (cf. Proposition 1.3.2). Here  $\lambda = \zeta_1 - 1$ , and  $\zeta_1$  is a primitive  $p$ -th root of 1.

Consider the Galois cover  $\mathcal{Y}'_1 \rightarrow \mathcal{P}_1$  which is generically given by the equation  $X^p = T^{-\alpha}(T^{-m} + \pi^{pt})$  where  $\alpha$  is an integer such that  $\alpha + m \equiv 0 \pmod{p}$ . Then  $\mathcal{Y}'_1$  is smooth over  $R$ , and the natural morphism  $\mathcal{Y}'_{1,k} \rightarrow \mathcal{P}_{1,k}$  between special fibres is radicial (cf. [Sa], Proposition 3.3.1, (b)). The above coverings can be patched using formal patching techniques to construct a Galois cover  $\tilde{\mathcal{Y}}_1 \rightarrow \mathcal{P}_1$  with Galois group  $G$  between semi-stable  $R$ -curves (cf. [Ga], and Proposition 1.2.2), and by construction the arithmetic genus  $g_1$  of the special fibre  $\tilde{\mathcal{Y}}_{1,k}$  [which is in fact equal to that of  $Y_k$ ] satisfies the inequality  $g_1 < g$  as required.  $\square$

**4.3.** Next, we will give some sufficient conditions for the existence of removable vertices in the case where the Galois group  $G \xrightarrow{\sim} \mathbb{Z}/p^2\mathbb{Z}$  has order  $p^2$ .

**Proposition 4.3.1.** *Assume that  $R$  contains a primitive  $p^2$ -th root of unity. Let  $f_k : Y_k \rightarrow \mathbb{P}_k^1$  be a finite ramified Galois cover with Galois group  $G \xrightarrow{\sim} \mathbb{Z}/p^2\mathbb{Z}$ , and  $Y_k$  a smooth  $k$ -curve. Let  $g_k : X_k \rightarrow \mathbb{P}_k^1$  be the Galois sub-cover of  $f_k$  with Galois group  $H \xrightarrow{\sim} \mathbb{Z}/p\mathbb{Z}$ , of cardinality  $p$ . Assume that there exists a smooth Galois lifting  $g : \mathcal{X} \rightarrow \mathbb{P}_R^1$  of  $g_k$  defined over  $R$  (cf. Definition 2.5.2). Assume that  $f_k$  satisfies the assumption **(A)** in 3.3.1 [with respect to the smooth lifting  $g$  of the Galois sub-cover  $g_k$ ]. Let  $\tilde{f} : \mathcal{Y} \rightarrow \mathbb{P}_R^1$  be a fake lifting of the Galois cover  $f_k : Y_k \rightarrow \mathbb{P}_k^1$  which dominates the smooth lifting  $g$  of  $g_k$  [which we suppose defined over  $R$ ] (cf. Definition 3.3.2). We assume that there exists a minimal birational morphism  $\mathcal{Y}' \rightarrow \mathcal{Y}$  with  $\mathcal{Y}'_k \stackrel{\text{def}}{=} \mathcal{Y}' \times_R k$  semi-stable, and such that the ramified points in the morphism  $f_K : \mathcal{Y}_K \rightarrow \mathbb{P}_K^1$  specialise in smooth distinct points of  $\mathcal{Y}'_k$ .*

Let  $\mathcal{P} \stackrel{\text{def}}{=} \mathcal{Y}'/G$  be the quotient of  $\mathcal{Y}'$  by  $G$  [which is a semi-stable  $R$ -model of  $\mathbb{P}_R^1$ ], and  $\Gamma''$  the tree which is associated to the special fibre  $\mathcal{P}_k$  of  $\mathcal{P}$ . Assume that there exists an internal vertex  $P_i$  of  $\Gamma''$  which satisfies the following properties.

- (i) The pre-image of  $P_i$  in  $\Gamma$  contains no ramified vertex.
- (ii) When moving in the tree  $\Gamma''$  from the vertex  $P_i$  towards the end vertices of  $\Gamma''$  we encounter a vertex [necessarily terminal by Theorem 3.5.4 (v)] whose pre-image in  $\Gamma$  contains a separable vertex.
- (iii) When moving in the tree  $\Gamma''$  from the vertex  $P_i$  towards the end vertices of  $\Gamma''$  we encounter a unique vertex whose pre-image in  $\Gamma$  contains [in fact consists of] a ramified vertex of type 2.
- (iv) When moving in the tree  $\Gamma''$  from the vertex  $P_i$  towards the end vertices of  $\Gamma''$  we encounter no vertex whose pre-image in  $\Gamma$  contains a ramified vertex of type 1.

Then  $P_i$  is a removable vertex of  $\Gamma''$  in the sense of Definition 4.1.2, and the revisited Oort conjecture [**Conj-O2-Rev**] is true for the Galois cover  $f_k : Y_k \rightarrow \mathbb{P}_k^1$ , and the smooth lifting  $g$  of the Galois sub-cover  $g_k$ .



*Proof.* Let  $Y_i$  be a vertex of the graph  $\Gamma$  which is in the pre-image of the vertex  $P_i$ , and  $D_i$  (resp.  $I_i$ ) the decomposition (resp. inertia) subgroup of the Galois group  $G$  at the generic point of  $Y_i$ . Then  $I_i \neq \{1\}$ , since the vertex  $Y_i$  is not terminal (cf. Theorem 3.5.4 (v)). Moreover,  $I_i = D_i = G$ , for otherwise we will contradict the assumption (iii) satisfied by  $P_i$  above.

Let  $\mathcal{P}_i$ ,  $\mathcal{P}'_i$ ,  $\mathcal{P}''_i$  and  $\mathcal{P}_{i,\infty}$  be as in 4.1.1. Let  $H' \subset G$  be the unique subgroup of  $G$  with cardinality  $p$ , and  $\mathcal{X}' \stackrel{\text{def}}{=} \mathcal{Y}'/H'$  the quotient of  $\mathcal{Y}'$  by  $H'$ . We have natural morphisms  $f' : \mathcal{Y}' \rightarrow \mathcal{X}' \rightarrow \mathcal{P}$ .

The Galois cover  $\mathcal{X}' \rightarrow \mathcal{P}$  induces above the irreducible component  $P_i$  of  $\mathcal{P}_k$ , outside the specialisation of the branched points, and the double points of  $\mathcal{P}$  supported by  $P_i$ , an  $\mathcal{H}_{pt,R}$ -torsor (cf. 1.3.1), where  $pt < v_K(\zeta_1 - 1)$ , and  $\zeta_1$  is a primitive  $p$ -th root of 1. This torsor is generically given by an equation

$$Z^p = 1 + \pi^{tp^2} g(T),$$

where  $1 + \pi^{tp^2} g(T) \in \text{Fr}(R < \frac{1}{T} >)$  has  $m + 1$  distinct geometric zeros in  $\mathcal{P}'_1$ , which we may assume without loss of generality specialise in the point  $\frac{1}{t} = 0$  at infinity [the later follows from the uniqueness of the ramified vertex of type 2 in the assumption (iii)]. We will assume for simplicity that  $g(T) = T^m$ . [The general case is treated in a similar fashion].

The above Galois cover  $f' : \mathcal{Y}' \rightarrow \mathcal{P}$  induces a cyclic Galois cover  $\mathcal{Y}'_\infty \rightarrow \mathcal{P}_{i,\infty}$  of degree  $p^2$  above the formal open annulus  $\mathcal{P}_{i,\infty}$ , with  $\mathcal{Y}'_\infty$  connected [since  $D_i = I_i = G$ ], which induces a cyclic Galois cover

$$f'_{\infty,1} : \mathcal{Y}'_{\infty,1} \rightarrow \mathcal{X}'_{\infty,1} \rightarrow \mathcal{P}_{i,\infty,1} \xrightarrow{\sim} \text{Spf } R[[T]]\{T^{-1}\}$$

of degree  $p^2$  above the formal boundary  $\mathcal{P}_{i,\infty,1} \xrightarrow{\sim} \text{Spf } R[[T]]\{T^{-1}\}$  of  $\mathcal{P}_{i,\infty}$  which is linked to  $\mathcal{P}'_i \xrightarrow{\sim} \text{Spf } R < \frac{1}{T} >$ . We will give an explicit description of the Galois cover  $f'_{\infty,1}$ , using the assumptions satisfied by the vertex  $P_i$ .

The Galois cover

$$\mathcal{X}'_{\infty,1} \rightarrow \mathcal{P}_{i,\infty,1}$$

is a torsor under the group scheme  $\mathcal{H}_{pt,R}$  [where  $t$  is as above] which has a degeneration type  $(\alpha_p, -m, 0)$ , where  $m > 1$  is as above [this results from the assumption (iii) satisfied by  $P_i$ ], and is given by an equation

$$(*)' \quad \frac{(\pi^{pt} X_1 + 1)^p - 1}{\pi^{p^2 t}} = T^m,$$

where  $pt < v_K(\zeta_1 - 1)$ , and  $\zeta_1$  is a primitive  $p$ -th root of 1 as above [in general replace  $T^m$  by  $g(T)$  above]. The  $\alpha_p$ -torsor

$$\mathcal{X}'_{\infty,1,k} \rightarrow \mathcal{P}_{i,\infty,1,k}$$

at the level of special fibres is given by the equation

$$x_1^p = t^m,$$

where  $x_1 = X_1 \pmod{\pi}$ , and  $t = T \pmod{\pi}$ .

From the above equation ( $*$ ) we deduce that in  $\mathcal{X}'_{\infty,1}$ , we have

$$T = (X_1^{\frac{1}{m}})^p [1 + \sum_{k=1}^{p-1} \binom{p}{k} \pi^{pt(k-p)} X_1^{k-p}]^{\frac{1}{m}}.$$

In particular,  $\mathcal{X}'_{\infty,1,k} \xrightarrow{\sim} \mathrm{Spf} R[[T_i]]\{T_i^{-1}\}$ , and  $X_1^{\frac{1}{m}}$  is a parameter of  $\mathcal{X}'_{\infty,1,k}$ .

Moreover, the Galois cover  $\mathcal{Y}'_{\infty,1} \rightarrow \mathcal{X}'_{\infty,1}$  is given by an equation

$$X_2^p = (1 + \pi^{pt} X_1)(1 + \pi^{ps} f(T))$$

where  $f(T) \in \mathrm{Fr}(R < \frac{1}{T} >)$  is such that  $(1 + \pi^{ps} f(T))$  is a unit in  $\mathcal{P}_i$ , for otherwise we will contradict the assumption (iv) satisfied by  $\mathcal{P}_i$ . We can assume without loss of generality that  $1 + \pi^{pt} f(T) \in R < \frac{1}{T} >$ . We will give an explicit description [by equations] of the degeneration of the Galois cover  $\mathcal{Y}'_{\infty,1} \rightarrow \mathcal{X}'_{\infty,1}$ .

Assume for simplicity that  $f(T) = T^{-m_1}$ , with  $m_1 > 0$ . The general case is treated in a similar fashion. Thus, our equation is

$$X_2^p = (1 + \pi^{pt} X_1)(1 + \pi^{ps} T^{-m_1}).$$

Assume first that  $t \leq s$ . Then on the level of special fibres the  $\alpha_p$ -torsor

$$\mathcal{Y}'_{\infty,1,k} \rightarrow \mathcal{X}'_{\infty,1,k}$$

is given [in the case where  $s = t$  one has to eliminate  $p$ -powers] by the equation

$$x_2^p = x_1 = (x_1^{\frac{1}{m}})^m$$

where  $x_1 = X_1 \bmod \pi$  ( $t^{-1}$  becomes a  $p$ -power in  $\mathcal{X}'_{\infty,1,k}$ ). In this case the above cover  $\mathcal{Y}'_{\infty,1} \rightarrow \mathcal{X}'_{\infty,1}$  is a torsor under the group scheme  $\mathcal{H}_{t,R}$ , and has a degeneration of type  $(\alpha_p, -m, 0)$ . [Note that  $x_1^{\frac{1}{m}}$  is a parameter of  $\mathcal{X}'_{\infty,1,k}$ ].

Assume now that  $s < t$ . Then

$$X_2^p = 1 + \pi^{ps} T^{-m_1} + \pi^{pt} X_1 + \pi^{p(t+s)} X_1 T^{-m_1},$$

which is not an integral equation for  $\mathcal{Y}'_{\infty,1}$ , since  $T^{-m_1}$  is a  $p$ -power mod  $\pi$  in  $\mathcal{X}'_{\infty,1,k}$ .

To obtain an integral equation we need first to replace  $T^{-m_1}$  by its expression, which is deduced from the above description of  $T$ ,

$$T^{-m_1} = (X_1^{\frac{1}{m}})^{-m_1 p} [1 + \sum_{k=1}^{p-1} \binom{p}{k} \pi^{pt(k-p)} X_1^{k-p}]^{\frac{-m_1}{m}}.$$

Thus,

$$X_2^p = 1 + \pi^{ps} (X_1^{\frac{1}{m}})^{-m_1 p} + \dots + \pi^{pt} X_1 + \dots,$$

where the remaining terms have coefficients with a valuation which is greater than  $ps$ . After replacing  $1 + \pi^{ps} (X_1^{\frac{1}{m}})^{-m_1 p}$  by

$$(1 + \pi^s (X_1^{\frac{1}{m}})^{-m_1})^p - \dots, \dots,$$

and multiplying the above equation by  $(1 + \pi^s(X_1^{\frac{1}{m}})^{-m_1})^{-p}$ , we reduce to an equation

$$(X'_2)^p = 1 + \pi^{pt}(X_1^{\frac{1}{m}})^m + \dots,$$

where the remaining terms have coefficients with a valuation which is greater than  $pt$ .

In particular, the Galois cover  $\mathcal{Y}'_{\infty,1} \rightarrow \mathcal{X}'_{\infty,1}$  is a torsor under the group scheme  $\mathcal{H}_{t,R}$  and has a degeneration of type  $(\alpha_p, -m, 0)$ . More precisely, the  $\alpha_p$ -torsor  $\mathcal{Y}'_{\infty,1,k} \rightarrow \mathcal{X}'_{\infty,1,x}$  on the level of special fibres is given by an equation

$$\tilde{x}_2^p = x_1 = (x_1^{\frac{1}{m}})^m.$$

The Galois cover  $f' : \mathcal{Y}' \rightarrow \mathcal{P}$  restricts to Galois covers  $\mathcal{Y}'_1 \rightarrow \mathcal{P}''_i$ , and  $\mathcal{Y}'_{1,\infty} \rightarrow \mathcal{P}'_{i,\infty}$ , above  $\mathcal{P}''_i$ , and  $\mathcal{P}_{i,\infty}$ , respectively. Consider the cyclic Galois cover  $\mathcal{Y}_1 \rightarrow \mathcal{X}_1 \rightarrow \mathcal{P}'_i$  of degree  $p^2$  which is generically given by the equations

$$\frac{(\pi^{pt}X_1 + 1)^p - 1}{\pi^{p^2t}} = T^m,$$

[in general replace  $T^m$  by  $g(T)$  above], and

$$X_2^p = (1 + \pi^{pt}X_1)(1 + \pi^{ps}f(T)),$$

where  $t$ ,  $s$ , and  $f(T)$  are as above. This Galois cover on the generic fibre is ramified only at ramified points of type 2  $[(1 + \pi^{ps}f(T))$  is a unit in  $\mathcal{P}'_i]$ . Furthermore, both  $\mathcal{X}_1$  and  $\mathcal{Y}_1$  are smooth, and the arithmetic genus of the special fibre  $\mathcal{Y}_{1,k}$  is 0. Indeed,  $\mathcal{X}_1$  is smooth, and the  $\alpha_p$ -torsor  $\mathcal{Y}_{1,k} \rightarrow \mathcal{X}_{1,k}$  is given by an equation  $\tilde{x}_2^p = x_1$  by arguments similar to the one above. [One also uses the fact that  $x_1^{\frac{1}{m}}$  is a parameter on  $\mathcal{X}_{1,k}$ ].

The above coverings can be patched using formal patching techniques to construct a Galois cover  $\tilde{\mathcal{Y}}_1 \rightarrow \mathcal{P}_i$  with Galois group  $G$  between semi-stable  $R$ -curves (cf. [Ga], and Proposition 1.2.2), and by construction the arithmetic genera  $g_1$  and  $g$  of the special fibres  $\tilde{\mathcal{Y}}_{1,k}$  and  $\mathcal{Y}'_k$  satisfy the inequality  $g_1 < g$ . Indeed, we have eliminated the contribution to the arithmetic genus of  $\mathcal{Y}'_k$  which arise from the separable end components of  $\Gamma$ , that lie above the end components of  $\Gamma''$  that we encounter when moving from the vertex  $P_i$  towards the ends of  $\Gamma''$ , and which exist by the assumption (ii) satisfied by  $P_i$ . This proves that  $P_i$  is a removable vertex as claimed.  $\square$

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